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Regularization of Toda lattices by Hamiltonian reduction

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Abstract

The Toda lattice defined by the Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \nu_i e^{q_i - q_{i+1}}$ with $\nu_i \in \{\pm 1\}$, which exhibits singular (blowing up) solutions if some of the $\nu_i = -1$, can be viewed as the reduced system following from a symmetry reduction of a subsystem of the free particle moving on the group $G = SL(n, \mathbb{R})$. The subsystem is T^*G_e, where $G_e = N_+AN_-$ consists of the determinant one matrices with positive principal minors, and the reduction is based on the maximal nilpotent group $N_+ \times N_-$. Using the Bruhat decomposition we show that the full reduced system obtained from T^*G, which is perfectly regular, contains 2^{n-1} Toda lattices. More precisely, if n is odd the reduced system contains all the possible Toda lattices having different signs for the ν_i. If n is even, there exist two non-isomorphic reduced systems with different constituent Toda lattices. The Toda lattices occupy non-intersecting open submanifolds in the reduced phase space, wherein they are regularized by being glued together. We find a model of the reduced phase space as a hypersurface in $\mathbb{R}^{2n-1}$. If $\nu_i = 1$ for all i, we prove for $n = 2, 3, 4$ that the Toda phase space associated with T^*G_e is a connected component of this hypersurface. The generalization of the construction for the other simple Lie groups is also presented.

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1. Introduction

The Toda lattice and its generalizations have been the subject of intense studies during the last three decades and are widely recognized as one of the most important families of models in the theory of integrable systems. Various Toda systems still attract attention today from viewpoints spanning from differential geometry to conformal field theory. The

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simplest of these systems is the open (non-periodic) finite Toda lattice, whose dynamics is generated by the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \nu_i e^{q_i - q_{i+1}} \qquad (\nu_i\colon \text{non-zero constant}) \tag{1.1}$$

on the phase space $\mathbb{R}^{2n}$ with the canonical Poisson brackets. This system, more precisely its center of mass reduction defined by setting $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 0$ and appropriately modifying the Poisson bracket, was historically the main source of the Adler–Kostant–Symes (AKS) theory of integrable Hamiltonian systems. In this Lie algebraic theory, whose most general form is based on the classical r-matrix, the phase spaces of integrable systems are realized as coadjoint orbits. See e.g. the reviews in [P,RSTS2] and references therein.

In this paper we study an aspect of the Toda lattices defined in (1.1), namely, their regularization in the singular case for which some of the real constants ν_i are negative. It is well known that the Hamiltonian vector field of the open Toda lattice is complete if all the constants ν_i are positive. In contrast, the Hamiltonian vector field is incomplete if $\nu_i < 0$ for some i. Indeed, if some $\nu_i < 0$ then it is intuitively clear that there should exist motions reaching infinity at finite time due to the rapidly decreasing exponential potential. Surprisingly, as far as we know, a proof confirming this expectation only appeared very recently [GS,KY].

Regularizing a singular, incomplete dynamical system means embedding it into a larger regular system whereby the incomplete trajectories get smoothly continued from $-\infty$ to $+\infty$ in time. Perhaps the most famous example is the regularization of the Kepler problem obtained by mapping the trajectories of negative energy into geodesics on a three-sphere S^3. The regularization of an incomplete Toda lattice will be achieved by realizing the system as a restriction of a larger Hamiltonian system having complete Hamiltonian vector field to an open submanifold of the phase space. The possibility of such a regularization was emphasized by Reyman and Semenov-Tian-Shansky [RSTS1,R,RSTS2]. Specifically, the idea is to apply Hamiltonian symmetry reduction to the system describing a free particle moving on a Lie group G, in our case $G = SL(n,\mathbb{R})$, in such a way that the reduced system is complete and contains the Toda lattice on an open submanifold. The way to define the symmetry reduction emerges naturally from the orbital AKS interpretation of the Toda lattice [RSTS1,R,RSTS2]. Despite the idea being known for quite a while, the completion of Toda lattices resulting from Hamiltonian reduction has not yet been studied in detail. In this paper it will be shown that the reduced phase space contains a *dense* open submanifold consisting of 2^{n-1} Toda lattices which are in general singular on their own but have their incomplete trajectories glued together "at infinity" represented by the complement of this submanifold. The full reduced system has an intricate structure, which we will try to explore.

Incidentally, our motivation originally comes from our earlier studies [FORTW] of the field theoretic version of the open Toda lattices in which the method of Hamiltonian reduction was used to derive them from the Wess–Zumino–Novikov–Witten model, which is a field theoretic generalization of a free particle on a group. In those investigations we realized that the reduced phase space automatically incorporates the singular solutions of the Toda field equations, which have received much attention in the simplest $SL(2,\mathbb{R})$ case of the Liouville equation [PP]. Our present study of Toda lattices with finitely many degrees of freedom may serve as a first step for field theoretic investigations in the same spirit. In fact, some remarks on this appeared already in our preliminary note in [TF] and a more extensive treatment in the Liouville case is in preparation [BFP].

The content of this paper is as follows: In Section 2 the orbital interpretation of the open Toda lattice is presented to fix the background for later sections. Two interpretations will be described in the AKS formalism in correspondence with two different splittings of the Lie algebra $sl(n,\mathbb{R})$. The first is the lower triangular–strictly upper triangular splitting which is relevant also in the "good sign" case, that is, the case of the Toda lattice with $\nu_i > 0$ for all i. The second is a lower triangular–pseudo-orthogonal splitting generalizing the lower triangular–orthogonal splitting used in the standard case. This second interpretation will allow us to make contact with the recent work in [KY]. The reader who is interested only in the regularization may skip this section.

Section 3 contains the symmetry reduction of the system on $T^*SL(n,\mathbb{R})$ that yields the regularization of the singular Toda lattices. Our definition of the reduction is slightly different from the one proposed in [RSTS1,R,RSTS2] since the strictly lower triangular $\times$ strictly upper triangular symmetry algebra is used instead of the lower triangular $\times$ strictly upper triangular one, but this leads to the same reduced phase space. Then the *Bruhat decomposition* of $SL(n,\mathbb{R})$, whose relevance to Toda lattices is well known [R,FH], is applied to describe the content of the reduced system. The reduced system turns out to contain 2^{n-1} Toda lattices [1] as subsystems with various signs in the Hamiltonian in correspondence with the non-intersecting open submanifolds of $SL(n,\mathbb{R})$ having non-vanishing principal minors with fixed signs. (For the precise statement, see the theorem of Section 3.) These Toda lattices are glued together along lower dimensional submanifolds in the phase space which are related to the submanifolds of $SL(n,\mathbb{R})$ with some vanishing principal minors.

Our aim in Sections 4–7 is to gain a better understanding of the reduced system. First we address the question of how many non-isomorphic possibilities are permitted by the reduction. There is a large freedom in choosing the value of the momentum map (which corresponds to a pair of matrices I_-, I_+ of the form in (2.2)) to fix the constraints, but there are also equivalences induced by the action of a certain group of diagonal matrices. In fact, we find that if n is odd then *all* reduced systems that arise are isomorphic Hamiltonian systems, and if n is even then there exist just *two* non-isomorphic reduced systems. This result is given by Proposition 1 in Section 4, and Proposition 2 describes the list of the Toda lattices contained in the non-isomorphic reduced systems.

In Section 5 an involutive symmetry of the reduced system, which derives from the outer automorphism of $sl(n,\mathbb{R})$, is exhibited. With respect to this symmetry, we establish the behaviour of the gauge invariant functions defined by the principal minors of $g \in SL(n,\mathbb{R})$.

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[1] In particular, a single Toda lattice does not form a dense submanifold in the reduced phase space, which seems to contradict a claim in [R].

The result, described in Propositions 3 and 4, proves to be useful when we analyse the reduced system.

Section 6 is devoted to deriving a model of the reduced phase space manifold, which is of dimension $2(n-1)$, in the form of a hypersurface in $\mathbb{R}^{2n-1}$. This is achieved with the aid of a global cross section of the gauge orbits in the constrained manifold of the reduction. We use an analogue of the "Drinfeld–Sokolov gauges" familiar in the context of n-KdV systems [DS]. As a result, the hypersurface is given by a polynomial equation. This model should be useful for further investigating the *topology* of the reduced phase space, although so far we have been able to carry this out only for some simple examples.

The examples alluded to in the above are contained in Section 7 and in Appendix A. In Section 7 the $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$ cases are presented in detail. In particular, in these cases the Toda lattice with good sign turns out to occupy a *connected component* in the full reduced phase space. It is appealing to conjecture that this is the case in general, since in the good sign case regularization is not needed as the Hamiltonian vector field is complete. This conjecture is verified in Appendix A for $SL(4,\mathbb{R})$, too.

In Section 8 a short discussion of the results and an outlook are offered. Finally, the generalization of the main construction of Section 3 is outlined for an arbitrary (real, split) simple Lie algebra in Appendix B.

2. Two orbital interpretations of the Toda lattice

We here explain two alternative orbital interpretations of the Toda lattice studied in this paper. For the Lie algebra $\mathcal{G} := sl(n,\mathbb{R})$, let us consider the decomposition

$$\mathcal{G} = \mathcal{G}_{<0} + \mathcal{G}_0 + \mathcal{G}_{>0} \tag{2.1}$$

defined by the subalgebras of strictly lower triangular, diagonal, and strictly upper triangular traceless matrices, respectively. Fix some elements

$$I_+ = \sum_{i=1}^{n-1} \nu_i^+ e_{i,i+1}, \qquad I_- = \sum_{i=1}^{n-1} \nu_i^- e_{i+1,i}, \tag{2.2}$$

where $e_{i,j}$ denotes the $n \times n$ elementary matrix having the entries $(e_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$ and $\nu_i^{\pm} \neq 0$ $(i = 1, \ldots, n-1)$ are some arbitrarily chosen real constants. The phase space of the Toda lattice, which we denote by M_e, is defined to be $M_e := \mathcal{G}_0 \times \mathcal{G}_0$. The general element of M_e is given by a pair, (q, p), of $n \times n$ diagonal, traceless matrices with real entries,

$$q = \mathrm{diag}\,(q_1, \ldots, q_n), \quad p = \mathrm{diag}\,(p_1, \ldots, p_n), \quad \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} p_i = 0. \tag{2.3}$$

The Toda dynamics is generated by the Hamiltonian H_e,

$$H_e(q,p) := \tfrac{1}{2}\mathrm{tr}(p^2) + \mathrm{tr}(I_- \mathrm{e}^q I_+ \mathrm{e}^{-q})$$

$$= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \nu_i e^{q_i - q_{i+1}}, \qquad \nu_i = \nu_i^- \nu_i^+, \tag{2.4}$$

by means of the symplectic structure $\omega_e = \mathrm{d}\,\mathrm{tr}(p\,\mathrm{d}q)$ on M_e leading to the Poisson brackets

$$\{q_i, q_k\} = \{p_i, p_k\} = 0, \quad \{q_i, p_k\} = \delta_{i,k} - \frac{1}{n}, \qquad i, k = 1, \ldots, n. \tag{2.5}$$

The corresponding equation of motion has the form

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = [I_-, e^q I_+ e^{-q}]. \tag{2.6}$$

The choice of $I_\pm$ in the above description of the system has a certain redundancy, since only the products $\nu_i = \nu_i^- \nu_i^+$ appear in the Hamiltonian. Moreover, using a constant shift of the coordinate q, we see that only the signs of the ν_i matter. Therefore, if desired, we could assume without loss of generality that, say,

$$\nu_i^- = 1, \qquad \nu_i^+ \in \{\pm 1\}. \tag{2.7}$$

If some of the ν_i in (2.4) are negative, then the Hamiltonian vector field in (2.6) is *incomplete*, which means that there exist trajectories blowing up to infinity at finite time. An interesting regularization of this singularity will be obtained from Hamiltonian reduction. Before discussing this, we wish to review the interpretation of the Toda lattice in the Adler–Kostant–Symes (AKS) framework (see [P,RSTS2] and references therein).

In the AKS approach to constructing integrable systems one starts with some Lie algebra $\mathcal{G}$ (in our case $\mathcal{G} = sl(n, \mathbb{R})$) and a splitting of $\mathcal{G}$ into a vector space direct sum of two Lie subalgebras $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{G} = \mathcal{A} + \mathcal{B}. \tag{2.8}$$

The dual space $\mathcal{G}^*$ has the induced decomposition

$$\mathcal{G}^* = \mathcal{A}^* + \mathcal{B}^*, \qquad \mathcal{A}^* = \mathcal{B}^\perp, \quad \mathcal{B}^* = \mathcal{A}^\perp, \tag{2.9}$$

where $\mathcal{A}^\perp \subset \mathcal{G}^*$ is the annihilator of $\mathcal{A} \subset \mathcal{G}$ with respect to the pairing $\langle\, ,\, \rangle : \mathcal{G}^* \times \mathcal{G} \to \mathbb{R}$. The trick is to endow the space $\mathcal{G}^*$ with the Poisson bracket $\{\, ,\, \}$ given by the direct difference of the Lie–Poisson brackets on $\mathcal{A}^*$ and $\mathcal{B}^*$, that is, for any smooth functions f, h on $\mathcal{G}^*$,

$$\{f, h\}(\xi) := \langle \xi, [\mathrm{d}f_{\mathcal{B}}(\xi), \mathrm{d}h_{\mathcal{B}}(\xi)]\rangle - \langle \xi, [\mathrm{d}f_{\mathcal{A}}(\xi), \mathrm{d}h_{\mathcal{A}}(\xi)]\rangle \quad \forall \xi \in \mathcal{G}^*, \tag{2.10a}$$

where we have decomposed the differential $\mathrm{d}f(\xi) \in \mathcal{G}$ of f as

$$\mathrm{d}f(\xi) = \mathrm{d}f_{\mathcal{A}}(\xi) + \mathrm{d}f_{\mathcal{B}}(\xi) \ \text{ with } \ \mathrm{d}f_{\mathcal{A}}(\xi) \in \mathcal{A},\ \mathrm{d}f_{\mathcal{B}}(\xi) \in \mathcal{B}. \tag{2.10b}$$

The symplectic leaves of the Poisson manifold $(\mathcal{G}^*, \{\, ,\, \})$ are the subspaces of the form

$$\mathcal{O} = \mathcal{O}_{\mathcal{A}} + \mathcal{O}_{\mathcal{B}}, \tag{2.11}$$

where $\mathcal{O}_{\mathcal{A}} \subset \mathcal{A}^*$ (resp. $\mathcal{O}_{\mathcal{B}} \subset \mathcal{B}^*$) is a coadjoint orbit corresponding to the action of $\mathcal{A}$ on $\mathcal{A}^*$ (resp. $\mathcal{B}$ on $\mathcal{B}^*$). Because of (2.10), here $\mathcal{A}^*$ (and thus also $\mathcal{O}_{\mathcal{A}}$) is endowed with its own Lie–Poisson bracket multiplied by (-1). The phase space $\mathcal{O}$ has a natural family of commuting Hamiltonians. These are given by the restrictions of the Casimir functions on $\mathcal{G}^*$, i.e., the elements of the ring $\mathcal{I}(\mathcal{G}^*)$ of functions invariant under the coadjoint action of the Lie algebra $\mathcal{G}$ on $\mathcal{G}^*$. The dynamics defined by $H \in \mathcal{I}(\mathcal{G}^*)$ has a generalized Lax form, since the Hamiltonian vector field χ_H on $\mathcal{G}^*$ generated by $H \in \mathcal{I}(\mathcal{G}^*)$ through the Poisson bracket in (2.10) is given by

$$\chi_H(\xi) = -\left(\mathrm{ad}^*\, \mathrm{d}H_{\mathcal{A}}(\xi)\right)(\xi) = \left(\mathrm{ad}^*\, \mathrm{d}H_{\mathcal{B}}(\xi)\right)(\xi) \qquad \forall \xi \in \mathcal{G}^*, \tag{2.12}$$

where $(\mathrm{ad}^* X)(\xi)$ stands for the coadjoint action of $X \in \mathcal{G}$ on $\xi \in \mathcal{G}^*$. The integration of Hamilton's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = \chi_H(\xi(t)) \tag{2.13}$$

can be reduced to a factorization problem in the connected Lie group G associated with $\mathcal{G}$. In fact [P,RSTS2], if one defines $X_0 := \mathrm{d}H(\xi_0)$ and considers the factorization

$$\mathrm{e}^{tX_0} = a(t)b(t), \tag{2.14}$$

where $a(t)$ (resp. $b(t)$) belongs to the Lie subgroup of G corresponding to the Lie subalgebra $\mathcal{A} \subset \mathcal{G}$ (resp. $\mathcal{B} \subset \mathcal{G}$), then one finds the solution of (2.13) through the initial value ξ_0 in the form

$$\xi(t) = \mathrm{Ad}^*_{a^{-1}(t)}\, \xi_0 = \mathrm{Ad}^*_{b(t)}\, \xi_0, \tag{2.15}$$

with Ad^*_g denoting the coadjoint action of $g \in G$ on $\mathcal{G}^*$.

In general the factorization problem (2.14) has a solution only locally, signalling the possible incompleteness of the Hamiltonian vector field, which is of course tangent to any orbit $\mathcal{O}$. Many interesting integrable systems can be described in the above framework. The interpretation of the Toda lattice presented in Section 2.1 will reappear in the subsequent Hamiltonian reduction treatment, while the alternative interpretation given in Section 2.2 allows us to make contact with the recent work in [KY].

2.1. *The lower triangular–strictly upper triangular splitting*

Let $\mathcal{G} := sl(n, \mathbb{R})$ and $G := SL(n, \mathbb{R})$. Consider the splitting (2.8) with $\mathcal{A}$ being the strictly upper triangular nilpotent subalgebra, and $\mathcal{B}$ the lower triangular Borel subalgebra,

$$\mathcal{A} := \mathcal{G}_{>0}, \qquad \mathcal{B} := \mathcal{G}_{\leq 0} = \mathcal{G}_0 + \mathcal{G}_{<0}. \tag{2.16}$$

Denote by N_+ and B_- the connected subgroups of G corresponding to $\mathcal{A}$ and $\mathcal{B}$. Identifying $\mathcal{G}^*$ with $\mathcal{G}$ using the scalar product provided by ordinary matrix trace, we have

$$\mathcal{A}^* = \mathcal{G}_{<0}, \qquad \mathcal{B}^* = \mathcal{G}_{\geq 0}. \tag{2.17}$$

Let us choose $\mathcal{O}_{\mathcal{A}}$ to be the one point coadjoint orbit (character) of N_+ given by $\mathcal{O}_{\mathcal{A}} = \{I_-\}$ with I_- in (2.2), and choose $\mathcal{O}_{\mathcal{B}}$ to be the coadjoint orbit of B_- through the point $I_+ \in \mathcal{B}^*$. The space $\mathcal{O} = \mathcal{O}_{\mathcal{A}} + \mathcal{O}_{\mathcal{B}}$ can then be parametrized as

$$\mathcal{O} = \left\{ J_e = I_- + p + \mathrm{e}^q I_+ \mathrm{e}^{-q} \mid \forall\, (q, p) \in \mathcal{G}_0 \times \mathcal{G}_0 \right\}. \tag{2.18}$$

We may identify $\mathcal{O}$ with the Toda phase space M_e defined above. The Hamiltonian H_e in (2.4) becomes

$$H_e(J_e) = \tfrac{1}{2}\mathrm{tr}(J_e^2). \tag{2.19}$$

A commuting family of independent Hamiltonians on $\mathcal{O}$ is provided by the set $H_i(J_e) = (1/i)\mathrm{tr}\left(J_e^i\right)$ for $i = 2, \ldots, n$. This implies the integrability of the Toda equation (2.6), which is generated by $H_e = H_2$ and can be re-casted according to (2.12) in the Lax form

$$\frac{\mathrm{d}J_e}{\mathrm{d}t} = [J_e\,,\, (J_e)_{>0}]. \tag{2.20}$$

According to (2.14) and (2.15), the integration algorithm corresponding to this orbital interpretation of the Toda lattice uses the Gauss decomposition

$$\mathrm{e}^{tJ_e} = n_+(t)b_-(t), \quad n_+(t) \in N_+, \;\; b_-(t) \in B_-, \tag{2.21}$$

which is valid if e^{tJ_e} belongs to a neighbourhood of the identity on the group $SL(n, \mathbb{R})$ (see Section 3).

2.2. *The lower triangular–pseudo-orthogonal splitting*

In the "good sign" case for which $\nu_i > 0$ the Toda vector field is known to be complete. This can be seen with the aid of the alternative orbital interpretation [P,RSTS2] given in terms of the Iwasawa decomposition of $\mathcal{G} = sl(n, \mathbb{R})$,

$$\mathcal{G} = \mathrm{o}(n, \mathbb{R}) + \mathcal{G}_{\leq 0}, \tag{2.22}$$

which corresponds to a global decomposition of G [H,W]. We below show that replacing $\nu_i > 0$ in the Toda Hamiltonian by $\nu_i \neq 0$ having arbitrary signs amounts to replacing $\mathrm{o}(n, \mathbb{R})$ in the splitting (2.22) by a pseudo-orthogonal Lie algebra. The singularity of the resulting Toda lattices is related to the fact that the pseudo-orthogonal analogue of the Iwasawa decomposition is not a global decomposition.

Given $I_\pm$ in (2.2), we can find a diagonal matrix $S = \mathrm{diag}\,(s_1, s_2, \ldots, s_n)$ for which

$$I_+ = S^{-1}(I_-)^{\mathrm{T}}S. \tag{2.23}$$

We use the notation X^{T} to denote the transpose of any matrix X. In fact, S in (2.23) is unique up to an overall constant, which may be fixed, e.g., by setting $s_1 = 1$. After choosing S, we define o_S to be the subalgebra of $sl(n, \mathbb{R})$ preserving the symmetric form on $\mathbb{R}^n$ associated with S,

$$o_S = \left\{ X \in sl(n, \mathbb{R}) \mid X^{\mathrm{T}} = -SXS^{-1} \right\}. \tag{2.24}$$

Up to isomorphism, the Lie algebra o_S depends only on the signature of the symmetric form associated with S. As is readily seen, a splitting of $\mathcal{G}$ is defined by

$$\mathcal{A} := o_S, \qquad \mathcal{B} := \mathcal{G}_{\leq 0}. \tag{2.25}$$

The space $\mathcal{B}^*$ in (2.9) is now identified as

$$\mathcal{B}^* = o_S^\perp = \left\{ L \in \mathcal{G} \mid L^{\mathrm{T}} = SLS^{-1} \right\}. \tag{2.26}$$

To re-obtain the Toda lattice, we now consider the coadjoint orbit $\tilde{\mathcal{O}}$ of the group B_- through the element $I := (I_- + I_+) \in \mathcal{B}^*$. This orbit can be parametrized as

$$\tilde{\mathcal{O}} = \left\{ L = \mathrm{e}^{q/2} I_+ \mathrm{e}^{-q/2} + p + \mathrm{e}^{-q/2} I_- \mathrm{e}^{q/2} \mid \forall (q,p) \in \mathcal{G}_0 \times \mathcal{G}_0 \right\}. \tag{2.27}$$

To compare with the space $\mathcal{O}$ in (2.18), we notice that the mapping

$$\phi : \mathcal{O} \to \tilde{\mathcal{O}}, \qquad \phi : J_e = (I_- + p + \mathrm{e}^{q} I_+ e^{-q}) \mapsto L = \mathrm{e}^{-q/2} J_e \mathrm{e}^{q/2}, \tag{2.28}$$

is a symplectomorphism if $\mathcal{O}$ is endowed with the Lie–Poisson bracket of $\mathcal{B}^*$ and $\tilde{\mathcal{O}}$ is by definition endowed with $\frac{1}{2}$-times the Lie–Poisson bracket of $\mathcal{B}^*$. In the $\tilde{\mathcal{O}}$ realization the commuting Hamiltonians are given by the trace of the powers of the Lax matrix L.

From (2.12) with the present conventions, the flow generated on $o_S^\perp$ by the Hamiltonian $H_k(L) = (1/k)\mathrm{tr}\left(L^k\right)$ takes the form

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tfrac{1}{2}[L\,,\,(L^{k-1})_{o_S}] \tag{2.29a}$$

with

$$(L^{k-1})_{o_S} = (L^{k-1})_{>0} - S^{-1}\left((L^{k-1})_{>0}\right)^{\mathrm{T}} S = (L^{k-1})_{>0} - (L^{k-1})_{<0}. \tag{2.29b}$$

The left hand side of this equation is defined using the splitting $\mathcal{G} = o_S + \mathcal{G}_{\leq 0}$ while the right hand side is defined using the decomposition of $\mathcal{G}$ in (2.1). One may check that the Toda equation (2.6) is recovered from (2.29) with $k = 2$ under restriction to the orbit $\tilde{\mathcal{O}}$.

The hierarchy on $o_S^\perp$ given in (2.29) is actually the same as the "Toda hierarchy with indefinite metric" introduced in [KY] generalizing the "full symmetric Toda hierarchy" studied in [DLNT] for which S is the identity matrix. This follows by noting that any Lax matrix $L \in o_S^\perp$ can be written in the form $L = lS$ with a symmetric matrix l, and this is precisely the form of the Lax matrix postulated in [KY]. In this reference the explicit solution of Eq. (2.29) with $k = 2$ was obtained in terms of the solution of a factorization problem involving the pseudo-orthogonal group whose Lie algebra is o_S. The above presentation permits us to view this factorization as a special case of the AKS scheme described earlier. More importantly, the results of [KY] (see also [GS]) show that if some $\nu_i < 0$ in (2.4) then the Toda system indeed has trajectories that blow up to infinity at finite time. To see this we recall that if $L_0 \in \tilde{\mathcal{O}}$ has a complex eigenvalue then the solution of the Toda equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tfrac{1}{2}\,[L\,,\,L_{>0} - L_{<0}] \tag{2.30}$$

through the initial value L_0 blows up to infinity (in q space) at finite time [KY,GS]. Clearly, if the energy $H_2 = \frac{1}{2}\mathrm{tr}(L^2)$ is negative then L must have a complex eigenvalue. On the other hand, since $\frac{1}{2}\mathrm{tr}(L^2) = H_e(q, p)$, we see from the formula of $H_e(q, p)$ in (2.4) that there exist initial data with negative energy whenever $\nu_i < 0$ for some i. Therefore in such a case the Toda lattice is singular [2] (incomplete). The blowing up of the solutions will be illustrated in examples later.

Remark 1. We can define a commuting hierarchy on $\mathcal{B}^*$ by restricting the Casimir functions on $\mathcal{G}^*$ to $\mathcal{B}^* \equiv \mathcal{A}^\perp + \{I\}$ using this identification for any choice of $\mathcal{A}$ in (2.8) and any character $I \in \mathcal{B}^\perp$ of $\mathcal{A}$. The resulting hierarchies are not isomorphic in general as illustrated by the hierarchies on $\mathcal{B}^* = (\mathcal{G}_{\leq 0})^*$ studied in [DLNT,KY]. Another hierarchy on $(\mathcal{G}_{\leq 0})^*$ is the "full Kostant–Toda lattice" investigated in [EFS], which is obtained by choosing $\mathcal{A} = \mathcal{G}_{>0}$ and $I := I_-$ in (2.2). Of course the restrictions of these hierarchies to some special coadjoint orbits in $\mathcal{B}^*$ can be (and are) isomorphic in some cases.

3. Regularization by Hamiltonian reduction

In this section we consider a Hamiltonian symmetry reduction of the system describing a free particle on the group $G = SL(n, \mathbb{R})$. This leads to a reduced system that contains the Toda lattice defined in the preceding section. More precisely, it turns out that the reduced system, which is perfectly regular, contains not only one but 2^{n-1} Toda lattices that are glued together in such a way to provide a natural regularization of their singularities. The idea that Hamiltonian reduction can be used to regularize singular, incomplete Toda systems arising in the AKS framework goes back to Reyman and Semenov-Tian-Shansky [RSTS1,R]. The details of this regularization has not been however worked out previously.

A model of the free particle on the group G is furnished by the Hamiltonian system $(\mathcal{M}, \Omega, \mathcal{H})$ as follows. The phase space $\mathcal{M}$ is the cotangent bundle of the group,

$$\mathcal{M} = T^*G \simeq \{(g, J) \mid g \in G,\ J \in \mathcal{G}\}, \tag{3.1}$$

where $\mathcal{G}^*$ is identified with $\mathcal{G}$ using the scalar product, and the identification $T^*G \simeq G \times \mathcal{G}$ is defined by right translations on G. The fundamental Poisson brackets are

$$\begin{gathered}\{g_{i,j}\,, g_{k,l}\} = 0, \qquad \{g_{i,j}\,, \mathrm{tr}(T^a J)\} = (T^a g)_{i,j}, \\ \{\mathrm{tr}(T^a J)\,, \mathrm{tr}(T^b J)\} = \mathrm{tr}([T^a, T^b] J),\end{gathered} \tag{3.2}$$

with T^a being a basis of $\mathcal{G} = sl(n, \mathbb{R})$. These derive from the symplectic form Ω,

$$\Omega = \, \mathrm{d}\,\mathrm{tr}(J\,\mathrm{d}g\, g^{-1}). \tag{3.3}$$

The Hamiltonian $\mathcal{H}$ is taken to be

$$\mathcal{H} = \tfrac{1}{2}\mathrm{tr}(J^2), \tag{3.4}$$

[2] The existence of a complex eigenvalue of L_0 is sufficient but not necessary [KY,GS] for the singularity of the solution. A necessary and sufficient condition is given in [GS].

which yields the dynamics,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \{g\,,\mathcal{H}\} = Jg, \qquad \frac{\mathrm{d}J}{\mathrm{d}t} = \{J\,,\mathcal{H}\} = 0. \tag{3.5}$$

Hence the particle, whose position is given by $g(t) \in G$, moves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}g}{\mathrm{d}t}g^{-1}\right) = 0. \tag{3.6}$$

Note that J is the infinitesimal generator, i.e., the momentum map, for the action of G on $\mathcal{M}$ defined by left translations, while the action of G defined by right translations is generated by the momentum map $\tilde{J} : \mathcal{M} \to \mathcal{G}$,

$$\tilde{J}(g, J) = -g^{-1}Jg. \tag{3.7}$$

We wish to consider symmetry reduction using the subgroup

$$N := N_+ \times N_- \tag{3.8a}$$

of the full symmetry group, where N_+ and N_- are the subgroups of G associated with the Lie algebras $\mathcal{G}_{>0}$ and $\mathcal{G}_{<0}$, respectively. The group N acts on $\mathcal{M}$ according to

$$(n_+, n_-) : (g, J) \mapsto (n_+ g n_-^{-1}, n_+ J n_+^{-1}) \quad \forall (n_+, n_-) \in N, \ \ (g, J) \in \mathcal{M}. \tag{3.8b}$$

Identifying the dual of the Lie algebra $\mathcal{N} = \mathcal{G}_{>0} \times \mathcal{G}_{<0}$ of N as $\mathcal{N}^* = \mathcal{G}_{<0} \times \mathcal{G}_{>0}$, the momentum map $\Phi : \mathcal{M} \to \mathcal{N}^*$ corresponding to the symmetry in (3.8) is given by

$$\Phi(g, J) = (J_{<0}, \tilde{J}_{>0}), \tag{3.9}$$

where $J = J_{<0} + J_0 + J_{>0}$ according to (2.1) and similarly for $\tilde{J}$. We define the symmetry reduction by fixing the value of the momentum map Φ to be $(I_-, -I_+) \in \mathcal{N}^*$, where $I_\pm$ are the matrices in (2.2). The reduced phase space is obtained as the factor space

$$\mathcal{M}^{\mathrm{red}}(I_-, I_+) = \mathcal{M}^{\mathrm{c}}(I_-, I_+)/N \tag{3.10}$$

with

$$\mathcal{M}^{\mathrm{c}}(I_-, I_+) := \Phi^{-1}(I_-, -I_+)\,. \tag{3.11a}$$

We shall often write simply $\mathcal{M}^{\mathrm{c}}$ and $\mathcal{M}^{\mathrm{red}}$ omitting the argument (I_-, I_+). In Dirac's terminology, the constraints defining $\mathcal{M}^{\mathrm{c}} \subset \mathcal{M}$,

$$J_{<0} = I_-, \qquad \tilde{J}_{>0} = -I_+, \tag{3.11b}$$

are *first class* and thus in (3.10) we have to factorize by the gauge group N. (In other words, $(I_-, -I_+) \in \mathcal{N}^*$ is a character.) It is known that $\mathcal{M}^{\mathrm{red}}$ is a smooth manifold since $(I_-, -I_+)$ is a regular value of the momentum map and the action of N on $\mathcal{M}^{\mathrm{c}}$ is free and proper [R]. Our task is to describe the reduced Hamiltonian system $(\mathcal{M}^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}}, \Omega^{\mathrm{red}})(I_-, I_+)$, where Ω^{red} and $\mathcal{H}^{\mathrm{red}}$ on $\mathcal{M}^{\mathrm{red}}$ are naturally inherited from Ω and $\mathcal{H}$ on $\mathcal{M}$.

Let $\boldsymbol{M}$ be the finite subgroup of $G = SL(n, \mathbb{R})$ consisting of diagonal matrices with entries ± 1. An element $m \in \boldsymbol{M}$ is a diagonal matrix

$$m = \operatorname{diag}(m_1, m_2, \ldots, m_n), \qquad m_i = \pm 1, \qquad \prod_{i=1}^{n} m_i = 1, \tag{3.12}$$

and hence $\boldsymbol{M}$ has 2^{n-1} elements. For any $m \in \boldsymbol{M}$, define $G_m \subset G$ by

$$G_m := N_+ m A N_-, \qquad A := \exp(\mathcal{G}_0), \tag{3.13}$$

where A is the subgroup of G consisting of diagonal matrices with positive entries. As a special case of the *Bruhat (Gelfand–Naimark) decomposition* of semisimple Lie groups [H,W], we have

$$G = \bigcup_m G_m \bigcup G_{\mathrm{low}} \quad \text{(disjoint union)}, \tag{3.14}$$

where the "big cell" $\bigcup_m G_m$ is an open, dense submanifold in G and G_{low} is the union of certain lower dimensional submanifolds of G. The decomposition $g = n_+ m a n_-$ of any $g \in G_m$, with $n_\pm \in N_\pm, a \in A$, is unique. The open submanifold $G_m \subset G$ is diffeomorphic to $N_+ \times A \times N_-$. The detailed structure of G_{low} will not be used in this paper.

The big cell $\bigcup_m G_m$ of $SL(n, \mathbb{R})$ is the open submanifold of determinant one matrices with non-vanishing principal minors. The element $m \in \boldsymbol{M}$ fixes the signs of the principal minors, which is possible in 2^{n-1} different ways. Correspondingly, G_{low} consists of the matrices with unit determinant and at least one vanishing principal minor.

Consider now the decomposition of $\mathcal{M} = T^*G$ induced by the Bruhat decomposition of G,

$$\mathcal{M} = \bigcup_m \mathcal{M}_m \cup \mathcal{M}_{\mathrm{low}} \qquad \text{(disjoint union)}. \tag{3.15}$$

The cells $\mathcal{M}_m = T^*G|_{G_m}$ and $\mathcal{M}_{\mathrm{low}} = T^*G|_{G_{\mathrm{low}}}$ in this partitioning of $\mathcal{M}$ are *invariant* submanifolds with respect to the action of the symmetry group $N_+ \times N_-$. Therefore the corresponding partitioning

$$\mathcal{M}^{\mathrm{c}} = \bigcup_m \mathcal{M}^{\mathrm{c}}_m \cup \mathcal{M}^{\mathrm{c}}_{\mathrm{low}} \tag{3.16}$$

of $\mathcal{M}^{\mathrm{c}}$ induces a decomposition of the reduced phase space,

$$\mathcal{M}^{\mathrm{red}} = \bigcup_m \mathcal{M}^{\mathrm{red}}_m \cup \mathcal{M}^{\mathrm{red}}_{\mathrm{low}} \quad \text{(disjoint union)}. \tag{3.17}$$

It is clear that $\mathcal{M}^{\mathrm{red}}_m$ is an open submanifold in $\mathcal{M}^{\mathrm{red}}$ for any $m \in \boldsymbol{M}$ and $\mathcal{M}^{\mathrm{red}}_{\mathrm{low}}$ is a union of lower dimensional submanifolds.

For I_+ and $m \in \boldsymbol{M}$ given, define

$$I_+^m := m I_+ m^{-1}. \tag{3.18}$$

The following theorem asserts that the subsystem

$$(\mathcal{M}^{\mathrm{red}}_m, \Omega^{\mathrm{red}}_m, \mathcal{H}^{\mathrm{red}}_m), \;\; m \in \boldsymbol{M}, \tag{3.19}$$

of the reduced Hamiltonian system $(\mathcal{M}^{\rm red}, \Omega^{\rm red}, \mathcal{H}^{\rm red})$ obtained by restriction to the submanifold $\mathcal{M}_m^{\rm red} \subset \mathcal{M}^{\rm red}$ is a Toda lattice of the type defined in Section 2.

Theorem. *The subsystem of the reduced system in* (3.19) *is a Toda lattice, whose phase space in the realization provided by the lower triangular–strictly upper triangular splitting is given by the space of Lax matrices J_m of the form*

$$J_m = I_- + p + \mathrm{e}^q I_+^m \mathrm{e}^{-q} \quad \forall\, (q, p) \in \mathcal{G}_0 \times \mathcal{G}_0. \tag{3.20}$$

Proof. From (3.11), the submanifold $\mathcal{M}_m^{\rm c} \subset \mathcal{M}^{\rm c}$ can be written as

$$\mathcal{M}_m^{\rm c} = \{\, (g, J) \mid g = n_+ m \mathrm{e}^q n_-,\ n_\pm \in N_\pm,\ q \in \mathcal{G}_0, \\ J_{<0} = I_-,\ (g^{-1} J g)_{>0} = I_+ \,\}. \tag{3.21}$$

Hence a model of $\mathcal{M}_m^{\rm red} = \mathcal{M}_m^{\rm c}/(N_+ \times N_-)$ is given by the local gauge section $\mathcal{C}_m \subset \mathcal{M}_m^{\rm c}$,

$$\mathcal{C}_m = \{\, (m\mathrm{e}^q, J) \mid q \in \mathcal{G}_0,\ J_{<0} = I_-,\ (\mathrm{e}^{-q} m^{-1} J m \mathrm{e}^q)_{>0} = I_+ \,\}. \tag{3.22}$$

Solving the condition in (3.22) for J yields that $J = J_m$ with J_m given in (3.20). Since $\mathcal{C}_m$ is a model of $\mathcal{M}_m^{\rm red}$, and since once J_m is given the values of both q and p are uniquely determined, we see that the manifold

$$\mathcal{O}_m := \left\{\, J_m = I_- + p + \mathrm{e}^q I_+^m \mathrm{e}^{-q} \mid \forall\, (q, p) \in \mathcal{G}_0 \times \mathcal{G}_0 \,\right\} \tag{3.23}$$

is an equally good model of the manifold $\mathcal{M}_m^{\rm red}$. We can regard $\mathcal{O}_m$ as a Toda orbit of the type in (2.18), where we simply replace I_+ by I_+^m. Upon the identification $\mathcal{O}_m \simeq \mathcal{M}_m^{\rm red}$, the symplectic structure on the coadjoint orbit $\mathcal{O}_m$ of $N_+ \times B_-$ is the same as the symplectic structure $\Omega_m^{\rm red}$ of $\mathcal{M}_m^{\rm red}$ following from the Hamiltonian reduction. In fact,

$$\Omega_m^{\rm red} = \mathrm{d}\, \mathrm{tr}\,(p\, \mathrm{d}q)\,. \tag{3.24}$$

The commuting Hamiltonians provided by the AKS scheme coincide with those induced by Hamiltonian reduction from the commuting Hamiltonians

$$\mathcal{H}_k(J) = \frac{1}{k} \mathrm{tr}(J^k), \quad k = 2, \ldots, n, \tag{3.25}$$

on $\mathcal{M}$. The proof of the theorem is completed by noting that the Toda Hamiltonian

$$\mathcal{H}_m^{\rm red}(J_m) = \tfrac{1}{2}\mathrm{tr}(J_m^2) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mu_i \mathrm{e}^{q_i - q_{i+1}}, \tag{3.26a}$$

where

$$\mu_i = m_i m_{i+1} \nu_i, \quad \nu_i = \nu_i^+ \nu_i^-, \tag{3.26b}$$

arises from (3.25) for $k = 2$. □

The Toda flow on the manifold $\mathcal{O}_m \simeq \mathcal{M}_m^{\rm red}$ is governed by the equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = [I_-, \mathrm{e}^q I_+^m \mathrm{e}^{-q}]. \tag{3.27}$$

This flow on $\mathcal{O}_m$ is incomplete (singular) if and only if the value of μ_i is negative for some i. But there is no singularity in the full reduced system. The incompleteness of the system on the coadjoint orbit $\mathcal{O}_m = \mathcal{M}_m^{\rm red}$ is a manifestation of the fact that the particle may leave the submanifold $\mathcal{M}_m^{\rm red} \subset \mathcal{M}^{\rm red}$ at finite time. In concrete terms, this simply means that the trajectory of the free particle on G determined by an initial value $(m\mathrm{e}^q, J_m)$ at $t = 0$, which is given explicitly by

$$g(t) = \mathrm{e}^{tJ_m} m \mathrm{e}^q, \tag{3.28}$$

may leave the open submanifold G_m. (Here we used the fact that the flow of the reduced system is obtained by just projecting the original flow on $\mathcal{M}^c \subset \mathcal{M}$ to $\mathcal{M}^{\rm red}$.) This happens at time t_* if a principal minor of $g(t_*)$ vanishes, which corresponds to q reaching infinity at $t = t_*$ from the perspective of $G_m \subset G$. In conclusion, the embedding of $\mathcal{O}_m \simeq \mathcal{M}_m^{\rm red}$ into $\mathcal{M}^{\rm red}$ for all $m \in \boldsymbol{M}$ provides a regularization of the singular Toda lattices whereby their blowing up trajectories are glued together "at infinity".

4. The set of non-isomorphic reduced systems

We have associated the reduced Hamiltonian system $(\mathcal{M}^{\rm red}, \Omega^{\rm red}, \mathcal{H}^{\rm red})$ with any fixed pair (I_-, I_+) of matrices given in (2.2). The set of reduced systems so obtained is apparently parametrized by the different choices of (I_-, I_+), but this parametrization is highly redundant since different choices may lead to isomorphic reduced systems. The aim of this section is to describe the non-isomorphic reduced systems obtained from $(\mathcal{M}, \Omega, \mathcal{H})$, $\mathcal{M} = T^*SL(n, \mathbb{R})$. The situation turns out to be different depending on whether n is odd or even. For n odd, we find that *all* reduced systems are isomorphic and contain a copy of the Toda lattice with Hamiltonian

$$\mathcal{H}_m^{\rm red}(q, p) := \tfrac{1}{2}\mathrm{tr}(p^2) + \mathrm{tr}\left(I_- \mathrm{e}^q I_+^m \mathrm{e}^{-q}\right) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mu_i \mathrm{e}^{q_i - q_{i+1}} \tag{4.1}$$

for any choice of $\mathrm{sign}(\mu_i)$ for $i = 1, \ldots, (n-1)$. For n even, there exist *two* non-isomorphic reduced systems whose constituent Toda lattices will be given by Proposition 2 later.

To search for the non-isomorphic reduced systems, let us consider a pair, $(D, \bar{D})$, of real, diagonal, non-singular matrices of equal determinant,

$$D = \mathrm{diag}\,(d_1, \ldots, d_n), \qquad \bar{D} = \mathrm{diag}\left(\bar{d}_1, \ldots, \bar{d}_n\right), \qquad \det D = \det \bar{D}. \tag{4.2}$$

Denote by Δ the group whose elements are such pairs, $(D_1, \bar{D}_1)(D_2, \bar{D}_2) = (D_1 D_2, \bar{D}_1 \bar{D}_2)$. To any $(D, \bar{D}) \in \Delta$, define $\phi_{(D,\bar{D})}$ to be the *symmetry transformation* of the Hamiltonian system $(\mathcal{M}, \Omega, \mathcal{H})$ given by

$$\phi_{(D,\bar{D})} : (g, J) \mapsto (Dg\bar{D}^{-1}, DJD^{-1}) \qquad \forall (g, J) \in \mathcal{M}. \tag{4.3}$$

Focusing on the constrained manifold $\mathcal{M}^c(I_-, I_+) \subset \mathcal{M}$, we see that $\phi_{(D,\bar{D})}$ in (4.3) induces an isomorphism

$$\phi_{(D,\bar{D})} : \mathcal{M}^c(I_-, I_+) \to \mathcal{M}^c(I_-^D, I_+^{\bar{D}}), \quad I_-^D = DI_-D^{-1}, \; I_+^{\bar{D}} = \bar{D}I_+\bar{D}^{-1}. \tag{4.4}$$

Because of

$$DN_+D^{-1} = N_+, \qquad \bar{D}N_-\bar{D}^{-1} = N_-, \tag{4.5}$$

$\phi_{(D,\bar{D})}$ in (4.4) maps gauge orbits to gauge orbits. Hence it gives rise to a mapping

$$\hat{\phi}_{(D,\bar{D})} : \mathcal{M}^{\rm red}(I_-, I_+) \to \mathcal{M}^{\rm red}(I_-^D, I_+^{\bar{D}}), \tag{4.6}$$

which is an isomorphism between the corresponding reduced systems.

In order to find the non-isomorphic reduced systems, we have to study the orbits of the group Δ acting on the set of matrices (I_-, I_+) as $\Delta \ni (D, \bar{D}) : (I_-, I_+) \mapsto (I_-^D, I_+^{\bar{D}})$.

Proposition 1. *If n is odd, then the set of pairs (I_-, I_+) of the form in (2.2) is a single orbit of the group Δ. If n is even, then the set of pairs (I_-, I_+) consists of two orbits of Δ.*

Proof. For arbitrarily fixed matrices,

$$\begin{aligned} I_- &= \sum_{i=1}^{n-1} \nu_i^- e_{i+1,i}, & I_+ &= \sum_{i=1}^{n-1} \nu_i^+ e_{i,i+1}, \\ \hat{I}_- &= \sum_{i=1}^{n-1} \hat{\nu}_i^- e_{i+1,i}, & \hat{I}_+ &= \sum_{i=1}^{n-1} \hat{\nu}_i^+ e_{i,i+1}, \end{aligned} \tag{4.7}$$

consider

$$\hat{I}_- = DI_-D^{-1}, \qquad \hat{I}_+ = \bar{D}I_+\bar{D}^{-1} \tag{4.8}$$

as an equation for $(D, \bar{D}) \in \Delta$. In detail, (4.8) requires

$$\hat{\nu}_i^- = d_{i+1}\nu_i^- d_i^{-1} \text{ and } \hat{\nu}_i^+ = \bar{d}_i \nu_i^+ \bar{d}_{i+1}^{-1} \qquad \forall i = 1, \ldots, (n-1), \tag{4.9}$$

which leads to

$$d_i = d_1 \prod_{l=1}^{i-1} \frac{\hat{\nu}_l^-}{\nu_l^-}, \quad \bar{d}_i = \bar{d}_1 \prod_{l=1}^{i-1} \frac{\nu_l^+}{\hat{\nu}_l^+} \qquad \forall i = 2, \ldots, n. \tag{4.10}$$

Introducing the notation

$$\nu^\pm := \prod_{i=1}^{n-1} \left(\nu_i^\pm\right)^{n-i}, \qquad \hat{\nu}^\pm := \prod_{i=1}^{n-1} \left(\hat{\nu}_i^\pm\right)^{n-i}, \tag{4.11}$$

it follows that

$$\det D = (d_1)^n \frac{\hat{\nu}^-}{\nu^-}, \qquad \det \bar{D} = (\bar{d}_1)^n \frac{\nu^+}{\hat{\nu}^+}. \tag{4.12}$$

If n is odd, a solution of (4.9) satisfying $\det D = \det \bar{D}$ can be always found, e.g. the solution with $\det D = \det \bar{D} = 1$ obtained by setting

$$d_1 := \sqrt[n]{\frac{\nu^-}{\hat{\nu}^-}}, \qquad \bar{d}_1 := \sqrt[n]{\frac{\hat{\nu}^+}{\nu^+}}. \tag{4.13}$$

Thus we have a single orbit of Δ in this case. If n is even, then

$$\frac{\det D}{\det \bar{D}} = \left(\frac{d_1}{\bar{d}_1}\right)^n \frac{\hat{\nu}^- \hat{\nu}^+}{\nu^- \nu_+} \tag{4.14}$$

can be set equal to 1 by choosing d_1 and $\bar{d}_1$ if and only if

$$\mathrm{sign}\left(\nu^- \nu^+\right) = \mathrm{sign}\left(\hat{\nu}^- \hat{\nu}^+\right). \tag{4.15}$$

Hence in this case there are two orbits of Δ in the space of pairs (I_-, I_+) in correspondence with $\mathrm{sign}\left(\nu^- \nu^+\right) = \pm$. □

For any integer $n > 1$, let us introduce the matrices

$$I_-^+ := \sum_{i=1}^{n-1} e_{i+1,i} \quad \text{and} \quad I_+^+ := \sum_{i=1}^{n-1} e_{i,i+1}, \quad I_+^- := \sum_{i=1}^{n-2} e_{i,i+1} - e_{n-1,n}. \tag{4.16}$$

It is clear from the proof of Proposition 1 that if n is odd, then any (I_-, I_+) can be mapped to (I_-^+, I_+^+) by the action of the group Δ. If n is even, then any (I_-, I_+) can be mapped either to (I_-^+, I_+^+) or to (I_-^+, I_+^-). Thus, for n even, we may choose

$$(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})_\pm := (\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})(I_-^+, I_+^\pm) \tag{4.17}$$

as representatives of the non-isomorphic reduced systems. For n odd, we shall take the system $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})(I_-^+, I_+^+)$ as the representative.

Referring to the decomposition in (3.17), we can now describe the set of Toda lattices contained in the reduced system as subsystems.

Proposition 2. *If n is odd, then $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})$ contains all the possible Toda lattices, having any signs for μ_i, $i = 1, \ldots, (n-1)$, in* (4.1). *If n is even, then the reduced system $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})_\sigma$, $\sigma = \pm$, contains those Toda lattices for which the constants μ_i defining the Hamiltonian in* (4.1) *satisfy*

$$\sigma = \mathrm{sign}\left(\prod_{i=1}^{n-1} \mu_i^{n-i}\right), \tag{4.18}$$

and each of these Toda lattices occur in two copies, on the submanifolds $\mathcal{M}^{\mathrm{red}}_{\pm m}$ for $\pm m \in \boldsymbol{M}$ in (3.17).

Proof. According to the theorem of the preceding section, the Bruhat decomposition of $SL(n, \mathbb{R})$ gives rise to the Toda lattices $\left(\mathcal{M}^{\mathrm{red}}_m, \Omega^{\mathrm{red}}_m, \mathcal{H}^{\mathrm{red}}_m\right) \subset (\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})$ having the Hamiltonian in (4.1) with $\mu_i = \nu_i^+ \nu_i^- m_i m_{i+1}$ for any $m \in \boldsymbol{M}$ (cf. (3.26)). Using the above chosen representatives of the non-isomorphic reduced systems, if n is odd one finds that $\mu_i = m_i m_{i+1}$. For arbitrarily given $\mu_i \in \{\pm 1\}$, this has a unique solution for $m \in \boldsymbol{M}$, which implies the claim. On the other hand, when n is even, one has $\mu_i = m_i m_{i+1}$ for $\sigma = +$ and $\mu_i = m_i m_{i+1}$ $(i \neq n-1)$, $\mu_{n-1} = -m_{n-1} m_n$ for $\sigma = -$. The claim then follows by means of a computation similar to that used in the proof of Proposition 1. □

Remark 2. For n even, the systems $(\mathcal{M}^{\text{red}}, \Omega^{\text{red}}, \mathcal{H}^{\text{red}})_+$ and $(\mathcal{M}^{\text{red}}, \Omega^{\text{red}}, \mathcal{H}^{\text{red}})_-$ are not isomorphic since only the former contains a Toda lattice whose Hamiltonian is bounded from below. Note also that the transformation given by $q_i \mapsto -q_{n+1-i}$, $p_i \mapsto -p_{n+1-i}$ for any i provides an isomorphism between the Toda lattices with respective Hamiltonians $H(q,p) = \frac{1}{2}\text{tr}(p^2) + \sum_{i=1}^{n-1} \mu_i e^{q_i - q_{i+1}}$ and $H(q,p) = \frac{1}{2}\text{tr}(p^2) + \sum_{i=1}^{n-1} \mu_{n-i} e^{q_i - q_{i+1}}$. (In the above we did not take this into account for counting the Toda lattice content of $\mathcal{M}^{\text{red}}$.)

5. An involutive symmetry of the reduced system

Below we exhibit an involutive symmetry of the reduced system, which is induced by a corresponding symmetry of the original system on $\mathcal{M} = T^*SL(n, \mathbb{R})$. In the final analysis, this symmetry is due to the reflection symmetry of the Dynkin diagram of the Lie algebra $sl(n, \mathbb{R})$, and it may be thought of as a global version of the symmetry mentioned at the end of Section 4. It will be used to simplify some arguments later in the paper.

Denote by X^τ the transpose of any $n \times n$ matrix X with respect to the *anti-diagonal*,

$$(X^\tau)_{i,j} := X_{n+1-j,n+1-i} \quad \forall\, 1 \le i, j \le n. \tag{5.1a}$$

This operation has similar properties as the usual transpose since

$$X^\tau = \eta X^{\mathrm{T}} \eta^{-1} \quad \text{with} \quad \eta_{i,j} := \delta_{i,n+1-i}. \tag{5.1b}$$

Then define the transformation $\varphi : \mathcal{M} \to \mathcal{M}$ by

$$\varphi : (g, J) \mapsto ((g^{-1})^\tau, -J^\tau) \quad \forall (g, J) \in \mathcal{M}. \tag{5.2}$$

Clearly, φ is an *involutive symmetry* of the Hamiltonian system $(\mathcal{M}, \Omega, \mathcal{H})$. On the constrained manifold defined in (3.11a), it acts according to

$$\varphi : \mathcal{M}^{\mathrm{c}}(I_-, I_+) \to \mathcal{M}^{\mathrm{c}}(-I_-^\tau, -I_+^\tau). \tag{5.3}$$

Since

$$n_\pm^\tau \in N_\pm \quad \text{for } n_\pm \in N_\pm\,, \tag{5.4}$$

φ in (5.3) maps gauge orbits to gauge orbits. Hence it naturally induces a mapping

$$\hat{\varphi} : \mathcal{M}^{\text{red}}(I_-, I_+) \to \mathcal{M}^{\text{red}}(-I_-^\tau, -I_+^\tau), \tag{5.5}$$

which is an *isomorphism* between the corresponding reduced systems. On the other hand, a straightforward calculation shows that the pairs (I_-, I_+) and $(-I_-^\tau, -I_+^\tau)$ can be transformed into each other by the action of the group Δ, i.e., there exist matrices $(D, \bar{D})$ of the form in (4.2) satisfying

$$I_- = D(-I_-^\tau)D^{-1}, \qquad I_+ = \bar{D}(-I_+^\tau)\bar{D}^{-1}. \tag{5.6}$$

The corresponding symmetry $\phi_{(D,\bar{D})}$ given in (4.3) induces the isomorphism

$$\hat{\phi}_{(D,\bar{D})} : \mathcal{M}^{\text{red}}(-I_-^\tau, -I_+^\tau) \to \mathcal{M}^{\text{red}}(I_-, I_+). \tag{5.7}$$

Composing the above two mappings, we have the involutive symmetry

$$\psi_{(D,\bar{D})} := \phi_{(D,\bar{D})} \circ \varphi : \mathcal{M} \to \mathcal{M} \tag{5.8a}$$

given explicitly as

$$\psi_{(D,\bar{D})} : (g, J) \mapsto (D(g^{-1})^\tau \bar{D}^{-1}, -DJ^\tau D^{-1}) \quad \forall (g, J) \in \mathcal{M}. \tag{5.8b}$$

The involution property, $\psi^2_{(D,\bar{D})} = \text{id}$, is easily verified and $\psi_{(D,\bar{D})}$ maps the constrained manifold $\mathcal{M}^c(I_-, I_+)$ to itself. The construction implies the following proposition.

Proposition 3. *Fix a pair (I_-, I_+) according to* (2.2). *Then $\psi_{(D,\bar{D})}$ given in* (5.8) *induces the mapping*

$$\hat{\psi}_{(D,\bar{D})} = \hat{\phi}_{(D,\bar{D})} \circ \hat{\varphi} : \mathcal{M}^{\text{red}}(I_-, I_+) \to \mathcal{M}^{\text{red}}(I_-, I_+), \tag{5.9}$$

which is an involutive symmetry of the reduced system $(\mathcal{M}^{\text{red}}, \Omega^{\text{red}}, \mathcal{H}^{\text{red}})(I_-, I_+)$.

It is worth noting that $\hat{\psi}_{(D,\bar{D})}$ permutes the Toda lattices associated with the Bruhat decomposition,

$$\hat{\psi}_{(D,\bar{D})} : \mathcal{M}^{\text{red}}_m \to \mathcal{M}^{\text{red}}_{m'} \quad \text{with} \quad m'_i = \text{sign}(d_i)\, m_{n+1-i}\, \text{sign}(\bar{d}_i), \tag{5.10}$$

where $d_i := D_{i,i}$ are the entries of the diagonal matrix D, and similarly for m, m' and $\bar{D}$. The mapping in (5.10) can be recognized to be just the isomorphism between the respective Toda lattices remarked in Section 4.

We now describe the symmetry transformation $\psi_{(D,\bar{D})}$ explicitly. We consider the case for which $I_- = I_-^+$, $I_+ = I_+^+$ in (4.16), since this will be used later. Defining the $n \times n$ diagonal matrix D_+ by

$$(D_+)_{i,j} := (-1)^i \delta_{i,j} \quad \forall\, 1 \le i, j \le n, \tag{5.11}$$

we may take

$$(D, \bar{D}) = (D_+, D_+) \tag{5.12}$$

as the solution of (5.6). Up to the trivial redefinition $(D, \bar{D}) \mapsto (-D, -\bar{D})$, which does not change $\psi_{(D,\bar{D})}$, this is the unique solution if n is odd. If n is even, then

$$(D, \bar{D}) = (D_+, -D_+) \tag{5.13}$$

is another solution. In this case, the composed transformation $\chi := \psi_{(D_+,D_+)} \circ \psi_{(D_+,-D_+)}$ simply operates as

$$\chi : (g, J) \mapsto (-g, J) \quad \forall (g, J) \in \mathcal{M}. \tag{5.14}$$

In particular, the symmetry $\hat{\chi} : \mathcal{M}^{\mathrm{red}} \to \mathcal{M}^{\mathrm{red}}$ induced by χ is responsible for the occurrence of the Toda lattices in the reduced system in two copies (cf. Proposition 2). This symmetry is available only for even n, because $\det(-g) = (-1)^n$.

Let us now consider the set of functions Q_i on $\mathcal{M} = G \times \mathcal{G}$ given by the principal minors of the matrix $g \in G = SL(n, \mathbb{R})$,

$$Q_i(g, J) := \bar{Q}_i(g), \qquad \bar{Q}_i(g) := \det \begin{pmatrix} g_{i,i} & \cdots & g_{i,n} \\ \vdots & & \vdots \\ g_{n,i} & \cdots & g_{n,n} \end{pmatrix} \quad \text{for} \quad i = 2, \ldots, n. \tag{5.15}$$

These functions are *invariant* under our gauge group $N_+ \times N_-$,

$$\bar{Q}_i(n_+ g n_-^{-1}) = \bar{Q}_i(g) \quad \forall n_\pm \in N_\pm, \ i = 2, \ldots, n. \tag{5.16}$$

On each connected component G_m of the big cell in (3.14), the signs of the functions $\bar{Q}_i$ are fixed according to

$$\operatorname{sign}\left(\bar{Q}_i\right)|_{G_m} = \operatorname{sign}\left(\prod_{k=i}^{n} m_k\right), \tag{5.17}$$

and these functions are in one-to-one correspondence with the components of q appearing in the decomposition $g = n_+ m e^q n_-$ for $g \in G_m$. The importance of the Q_i is that, unlike the components of q, they give rise to globally defined smooth functions [3] on $\mathcal{M}^{\mathrm{red}}$.

For later reference, we present the behaviour of the Q_i with respect to the symmetry transformation $\psi_{(D_+,D_+)}$ given above.

Proposition 4. *Using the above notations, we have* $Q_i \circ \psi_{(D_+,D_+)} = Q_{n+2-i}$ *for any* $i = 2, \ldots, n$, *that is,*

$$\bar{Q}_i(D_+(g^{-1})^\tau D_+^{-1}) = \bar{Q}_{n+2-i}(g) \quad \forall g \in SL(n, \mathbb{R}). \tag{5.18}$$

Proof. The equality $(D_+ g D_+^{-1})_{i,j} = (-1)^{i+j} g_{i,j}$ implies that $\bar{Q}_i(D_+ g D_+^{-1}) = Q_i(g)$ for any $i = 2, \ldots, n$. Hence it is enough to show that

$$\bar{Q}_i((g^{-1})^\tau) = \bar{Q}_{n+2-i}(g) \quad \forall g \in SL(n, \mathbb{R}). \tag{5.19}$$

For an element $g = n_+ m e^q n_-$ of the big cell of $SL(n, \mathbb{R})$, because of (5.16), (5.19) is equivalent to

[3] Intuitively, the functions on $\mathcal{M}^{\mathrm{red}}$ corresponding to the Q_i can be thought of as global "position" variables. But one must be careful since it appears (see Sections 6 and 7) that in general $\mathcal{M}^{\mathrm{red}}$ is *not* a cotangent bundle of some configuration space.

$$\bar{Q}_i(e^{-q^\tau}m^\tau) = \bar{Q}_{n+2-i}(me^q), \tag{5.20}$$

which is readily verified. This completes the proof, since the big cell is a dense submanifold of $SL(n, \mathbb{R})$. □

To summarize, at this stage we have a clear understanding of the Toda lattice content of the non-equivalent reduced systems and their residual discrete symmetries. Next we take some initial steps towards investigating the global structure of these systems.

6. A hypersurface model of $\mathcal{M}^{\mathrm{red}}$ from global gauge fixing

In Section 3 we used a *locally* – over the big cell of $SL(n, \mathbb{R})$ – valid gauge fixing to identify the reduced system $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})$ obtained from $T^*SL(n, \mathbb{R})$ as one containing 2^{n-1} Toda lattices glued together along lower dimensional submanifolds. To furnish a tool for studying the topology of $\mathcal{M}^{\mathrm{red}} = \mathcal{M}^{\mathrm{c}}/N$, we here describe a *global* cross section of the gauge orbits in $\mathcal{M}^{\mathrm{c}}$. The global gauge fixing permits us to find a model of the manifold $\mathcal{M}^{\mathrm{red}}$ in the form of a hypersurface in $\mathbb{R}^{2n-1}$.

The constraints on J given in (3.11) are well studied in the context of generalized KdV equations and $\mathcal{W}$-algebras. Drinfeld and Sokolov has shown in [DS] how to define a global gauge fixing for the gauge transformations generated by these constraints. The corresponding gauges are called "DS gauges", e.g., in [FORTW]. To obtain a global gauge fixing for the constraints in (3.11), we simply have to restrict both J and $\tilde{J}$ to a DS gauge. This has to be formulated in terms of the variables (g, J) since our phase space is not $\mathcal{G} \times \mathcal{G}$ but $\mathcal{M} = T^*G \simeq G \times \mathcal{G}$, with $G = SL(n, \mathbb{R})$ and $\mathcal{G} = sl(n, \mathbb{R})$. The so obtained global cross section of the N-orbits in $\mathcal{M}^{\mathrm{c}}$ will be henceforth called a "double DS gauge" (see also [TF]).

A double DS gauge can be described as follows. Let $V \subset \mathcal{G}_{\geq 0}$ and $\tilde{V} \subset \mathcal{G}_{\leq 0}$, where we use the principal grading of $\mathcal{G}$ as in (2.1), be graded linear subspaces[4] appearing in a linear direct sum decomposition

$$\mathcal{G}_{\geq 0} = [I_-, \mathcal{G}_{>0}] + V, \qquad \mathcal{G}_{\leq 0} = [I_+, \mathcal{G}_{<0}] + \tilde{V}. \tag{6.1}$$

Then define $\mathcal{M}^{\mathrm{DS}} \subset \mathcal{M}^{\mathrm{c}}$ as

$$\mathcal{M}^{\mathrm{DS}} = \{\, (g, J) \in \mathcal{M}^{\mathrm{c}} \mid J \in (I_- + V),\ (g^{-1}Jg) \in (I_+ + \tilde{V})\,\}. \tag{6.2}$$

The point is that this is a global cross section of the gauge orbits for any choice of V, $\tilde{V}$, and hence we may identify $\mathcal{M}^{\mathrm{red}} = \mathcal{M}^{\mathrm{c}}/N = \mathcal{M}^{\mathrm{DS}}$. Using (2.2), a convenient choice of V, $\tilde{V}$ is furnished by

$$V := \mathrm{span}\{\, e_{1,i} \text{ for } i = 2, \ldots, n\,\}, \qquad \tilde{V} := \mathrm{span}\{\, e_{i,1} \text{ for } i = 2, \ldots, n\,\}. \tag{6.3}$$

[4] That is, V and $\tilde{V}$ are invariant subspaces of ad I_0 for $2I_0 = \mathrm{diag}\,(n-1, n-3, \ldots, 1-n)$.

Corresponding to this choice, $(g, J) \in \mathcal{M}^{\mathrm{DS}}$ is restricted to satisfy

$$J = I_- + \sum_{i=2}^{n} u_i e_{1,i}, \quad g^{-1} J g = I_+ + \sum_{i=2}^{n} v_i e_{i,1}, \qquad \text{with some } u_i, v_i \in \mathbb{R}. \tag{6.4}$$

Notice that the parameters v_i can be determined in terms of the u_i from the relations

$$\operatorname{tr} J^k = \operatorname{tr}(g^{-1} J g)^k \quad \forall k = 2, \ldots, n. \tag{6.5}$$

Our problem is to solve these relations and to find a proper parametrization of $(g, J) \in \mathcal{M}^{\mathrm{DS}}$.

The solution of (6.5) depends on the choice of (I_-, I_+). Choosing the representatives in (4.16), we first treat the system

$$(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})(I_-^+, I_+^+). \tag{6.6}$$

From (6.4), (6.5) and $(I_+^+)^{\mathrm{T}} = I_-^+$, we have

$$\begin{aligned} \operatorname{tr} J^k &= \operatorname{tr}\Big(I_-^+ + \sum_{i=2}^{n} u_i e_{1,i}\Big)^k = \operatorname{tr}(g^{-1} J g)^k = \operatorname{tr}((g^{-1} J g)^{\mathrm{T}})^k \\ &= \operatorname{tr}\Big(I_-^+ + \sum_{i=2}^{n} v_i e_{1,i}\Big)^k, \end{aligned} \tag{6.7}$$

which implies

$$v_i = u_i \quad \forall i = 2, \ldots, n. \tag{6.8}$$

Then the second relation in (6.4) becomes

$$\left(I_-^+ + \sum_{i=2}^{n} u_i e_{1,i}\right) g = g \left(I_+^+ + \sum_{i=2}^{n} u_i e_{i,1}\right). \tag{6.9}$$

It follows from (6.9) that the matrix g has the same value along each anti-diagonal line,

$$g_{i,j} = g_{k,l} \qquad \text{if} \quad i + j = k + l. \tag{6.10}$$

It also follows that

$$g_{1,j-1} = \sum_{i=2}^{n} u_i g_{i,j} \quad \forall j = 2, \ldots, n, \tag{6.11}$$

which shows that all the "lower" entries $g_{1,j-1}$ in the first row for $1 \le j-1 \le n-1$ can be expressed in terms of the "higher" ones together with the u_i for $2 \le i \le n$. Thus, if we set

$$u_{i+j} = g_{i,j} \quad \text{for} \quad n+1 \le i+j \le 2n, \tag{6.12}$$

then in the double DS gauge the matrix g can be written as

$$g = \begin{pmatrix} g_{1,1}(u) & g_{1,2}(u) & \dots & g_{1,n-1}(u) & u_{n+1} \\ g_{1,2}(u) & g_{1,3}(u) & \dots & u_{n+1} & u_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{1,n-1}(u) & u_{n+1} & \dots & u_{2n-2} & u_{2n-1} \\ u_{n+1} & u_{n+2} & \dots & u_{2n-1} & u_{2n}. \end{pmatrix}. \tag{6.13}$$

In this way the set of variables $(u_2, u_3, \dots, u_{2n})$ provides a parametrization of $(g, J) \in \mathcal{M}^{\mathrm{DS}}$. These $(2n-1)$ parameters are subject to the relation $\det g(u) = 1$ in accordance with the fact that the dimension of $\mathcal{M}^{\mathrm{c}}/N$ is $2(n-1)$.

Remark 3. For any choice of V, $\tilde{V}$ in (6.1), the double DS gauge $\mathcal{M}^{\mathrm{DS}} \subset \mathcal{M}$ in (6.2) is invariant with respect to the original dynamics on $\mathcal{M}$ since $\mathcal{M}^{\mathrm{DS}}$ is defined by constraining J and $\tilde{J}$, which are constants of motion. Using the model $\mathcal{M}^{\mathrm{red}} = \mathcal{M}^{\mathrm{DS}}$, the trajectory $(g(t), J(t)) \in \mathcal{M}^{\mathrm{DS}}$ of the reduced Hamiltonian system associated with the initial value $(g, J) \in \mathcal{M}^{\mathrm{DS}}$ at $t = 0$ is given by

$$J(t) = J, \qquad g(t) = \mathrm{e}^{tJ} g. \tag{6.14}$$

For our choice of V, $\tilde{V}$ in (6.3), the evolution equation, $\mathrm{d}J(t)/\mathrm{d}t = 0$, $\mathrm{d}g(t)/\mathrm{d}t = J(t)g(t)$, of the reduced system in (6.6) can be re-casted as

$$\begin{aligned} &\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = 0 \quad \forall i = 2, \dots, n, \\ &u_k(t) = \frac{\mathrm{d}^{2n-k} u_{2n}(t)}{\mathrm{d}t^{2n-k}} \quad \forall k = n+1, \dots, 2n-1, \end{aligned} \tag{6.15a}$$

and

$$\frac{\mathrm{d}^n u_{2n}(t)}{\mathrm{d}t^n} - \sum_{i=2}^{n} u_i \frac{\mathrm{d}^{n-i} u_{2n}(t)}{\mathrm{d}t^{n-i}} = 0. \tag{6.15b}$$

Solving this linear differential equation with *constant* coefficients for a given initial value $(u_2, \dots, u_{2n})$, subject to $\det g(u) = 1$ with $g(u)$ in (6.13), is equivalent to computing the last row of the matrix e^{tJ} in (6.14).

We wish to present a reinterpretation of the above identification $\mathcal{M}^{\mathrm{red}} = \mathcal{M}^{\mathrm{DS}}$. For this let us now think of $u := (u_2, u_3, \dots, u_{2n})$ as the general element of the manifold $\mathbb{R}^{2n-1}$ and define the polynomial P on $\mathbb{R}^{2n-1}$ by

$$P : u \mapsto \det g(u), \tag{6.16}$$

where $g(u)$ is determined by (6.13) using (6.10), (6.11). We then define

$$\mathcal{S}_+(n) := \{ u \in \mathbb{R}^{2n-1} \mid P(u) = 1 \}. \tag{6.17}$$

Computations made for small n (see Section 7) support the following *conjecture*:

$$\mathrm{d}P(u)|_{P(u)=1} \neq 0. \tag{6.18}$$

If (6.18) is valid, then $\mathcal{S}_+(n) \subset \mathbb{R}^{2n-1}$ is a *regular* submanifold. On the other hand, we can consider the mapping

$$(u_2, u_3, \ldots, u_{2n}) : \mathcal{M}^{\mathrm{DS}} \to \mathbb{R}^{2n-1} \tag{6.19}$$

engendered by the above parametrization of $(g, J) \in \mathcal{M}^{\mathrm{DS}}$. This mapping is smooth by construction. In fact, it follows from DS gauge fixing that $u_2, u_3, \ldots, u_{2n}$ are given by polynomials when regarded as gauge invariant functions on $\mathcal{M}^{\mathrm{c}}$. It is also a tautological statement that the mapping in (6.19) is a one-to-one mapping from $\mathcal{M}^{\mathrm{DS}}$ to its image given by $\mathcal{S}_+(n) \subset \mathbb{R}^{2n-1}$. If $\mathcal{S}_+(n) \subset \mathbb{R}^{2n-1}$ is a regular submanifold, then we can conclude from a well-known theorem in differential geometry that the mapping in (6.19) gives rise to a *diffeomorphism*

$$(u_2, u_3, \ldots, u_{2n}) : \mathcal{M}^{\mathrm{DS}} \to \mathcal{S}_+(n). \tag{6.20}$$

This is a non-trivial statement since the manifold structure of $\mathcal{M}^{\mathrm{DS}} = \mathcal{M}^{\mathrm{c}}/N$ is determined by the reduction and that of $\mathcal{S}_+(n)$ is determined by its embedding into $\mathbb{R}^{2n-1}$.

In conclusion, modulo the conjecture given by (6.18), we have shown that *the manifold $\mathcal{M}^{\mathrm{red}}(I_-^+, I_+^+)$ is diffeomorphic to the regular hypersurface $\mathcal{S}_+(n) \subset \mathbb{R}^{2n-1}$ for any $n > 1$.*

We can analogously treat the system $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}_{\mathrm{red}})_-$ defined for any even n by taking $(I_-, I_+) = (I_-^+, I_+^-)$ in (4.17). Making use of the relation

$$I_+^- = D_0 I_+^+ D_0^{-1} \quad \text{with } D_0 = \mathrm{diag}(1, \ldots, 1, -1), \tag{6.21}$$

instead of (6.8) we find

$$v_i = u_i \quad \text{for } i = 2, \ldots, (n-1) \text{ and } v_n = -u_n. \tag{6.22}$$

Denoting the general element of $\mathcal{M}^{\mathrm{DS}}$ now as $(\bar{g}, J) \in \mathcal{M}^{\mathrm{DS}}$, it turns out that

$$\bar{g} = g D_0, \tag{6.23}$$

where the matrix $g = g(u)$ is still given by the same equations (6.10)–(6.13). The only difference is that in this case, due to (6.23), the restriction on the parameters $u = (u_2, u_3, \ldots, u_{2n})$ reads

$$P(u) = \det g(u) = -1. \tag{6.24}$$

Similarly to (6.18), we have the *conjecture*:

$$\mathrm{d}P(u)|_{P(u)=-1} \neq 0. \tag{6.25}$$

If this holds, then $\mathcal{M}^{\mathrm{DS}}$ is diffeomorphic to the regular hypersurface

$$\mathcal{S}_-(n) := \{\, u \in \mathbb{R}^{2n-1} \mid P(u) = -1 \,\}. \tag{6.26}$$

The regularity conjecture of Eqs (6.18), (6.25) will be seen to hold for the examples of the next section.

7. Examples

Below we develop the simplest examples of the construction of the preceding sections by studying the reduced systems $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})$ resulting from $T^*SL(n, \mathbb{R})$ for $n = 2, 3$. The conjectures in (6.18), (6.25) will be verified for these examples, which will illustrate the regularization of the singular Toda lattices. We shall also show that the Toda lattice having the Hamiltonian $H(q, p) = \frac{1}{2}\mathrm{tr}(p^2) + \sum_i \mathrm{e}^{q_i - q_{i+1}}$, for which regularization is not necessary since the Hamiltonian vector field is complete, is realized as a *connected component* of the reduced phase space. We conjecture that this Toda lattice with good sign – which is contained in $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})(I_-^+, I_+^+)$ according to Proposition 2 – is always topologically disconnected from the rest of the reduced phase space. It would be interesting to prove (or disprove) this conjecture for general n, but with our "direct inspection" method this is difficult as can be seen from Appendix A, where the conjecture is verified for $n = 4$.

7.1. The case of $SL(2, \mathbb{R})$

We wish to treat both systems

$$(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})_\sigma \quad \text{for} \quad \sigma = \pm \tag{7.1}$$

defined in (4.17). The matrices $J(u)$ of (6.4) and $g(u)$ of (6.12) can be written

$$J(u) = \begin{pmatrix} 0 & u_2 \\ 1 & 0 \end{pmatrix}, \qquad g(u) = \begin{pmatrix} u_2 u_4 & u_3 \\ u_3 & u_4 \end{pmatrix}. \tag{7.2}$$

Hence the equation of the hypersurface $\mathcal{S}_\sigma(2) \subset \mathbb{R}^3$ is given by

$$P(u) = \det g(u) = u_2 u_4^2 - u_3^2 = \sigma 1. \tag{7.3}$$

It is straightforward to check that

$$\mathrm{d}P(u)|_{P(u)=\sigma 1} \neq 0, \tag{7.4}$$

which implies that $\mathcal{S}_\sigma(2) \subset \mathbb{R}^3$ is a regular hypersurface diffeomorphic to $\mathcal{M}_\sigma^{\mathrm{red}}$. Regarding the u_i as gauge invariant functions on the constrained manifold $\mathcal{M}_\sigma^{\mathrm{c}}$, we find the Poisson brackets

$$\{u_4, u_3\} = \frac{u_4^2}{2}, \qquad \{u_4, u_2\} = u_3, \qquad \{u_3, u_2\} = u_4 u_2. \tag{7.5}$$

This fixes the reduced dynamics since the Hamiltonian is simply

$$\mathcal{H}_\sigma^{\mathrm{red}}(u) = \tfrac{1}{2}\mathrm{tr} J^2(u) = u_2. \tag{7.6}$$

Remark 4. One may use (7.5) to define a Poisson bracket on $\mathbb{R}^3$ with coordinates (u_2, u_3, u_4). In this case $P(u)$ in (7.3) acquires the interpretation of a Casimir function on the Poisson manifold $(\mathbb{R}^3, \{\ ,\ \})$. Then $\mathcal{S}_\pm(2) \subset \mathbb{R}^3$ are symplectic leaves.

In the $\sigma = +$ case Eq. (7.3) excludes the value $u_4 = 0$ and the surface $\mathcal{S}_+(2)$ is easily seen to be the union of two disconnected pieces, which are diffeomorphic to the plane as they can be parametrized respectively by $(u_3, u_4) \in \mathbb{R} \times \mathbb{R}^+$ and by $(u_3, u_4) \in \mathbb{R} \times \mathbb{R}^-$. On these two coordinate patches we can introduce new coordinates $(x_\pm, \pi_\pm) \in \mathbb{R}^2$ by

$$u_4 = \pm \exp(-x_\pm), \qquad u_3 = -\pi_\pm u_4, \tag{7.7}$$

in terms of which the reduced Poisson brackets is given by $\{x_\pm, \pi_\pm\} = 1/2$ and the reduced Hamiltonian, $\mathcal{H}_+^{\mathrm{red}}$, is written as

$$\mathcal{H}_+^{\mathrm{red}}(x_\pm, \pi_\pm) = \pi_\pm^2 + \exp(2x_\pm). \tag{7.8}$$

These relations take the form of Eqs. (2.4) and (2.5) in terms of the respective variables $q_\pm = \mathrm{diag}(x_+, -x_+)$ and $p_\pm = \mathrm{diag}(\pi_+, -\pi_+)$.

In the above we have decomposed the reduced system into the disconnected union of two Toda lattices with good sign. Of course these subsystems arise from the components of the big cell of $SL(2, \mathbb{R})$ containing $\pm e$. Since u_4 is nothing but the gauge invariant function of $(g, J) \in \mathcal{M}^c$ given by the matrix element g_{22}, the inequality $u_4 = g_{22} \neq 0$ excludes the part of $T^*SL(2, \mathbb{R})$ that does not lie over the big cell. This is the reason for the disconnectedness of $\mathcal{M}_{\sigma=+}^{\mathrm{red}} = \mathcal{S}_+(2)$.

Let us now discuss the case of $\sigma = -$. This is drastically different from the previous case, because Eq. (7.3) no longer excludes the value $u_4 = 0$ corresponding to the complement of the big cell of $SL(2, \mathbb{R})$. We still have two Toda lattices as subsystems for $u_4 > 0$ and for $u_4 < 0$, but now the reduced Hamiltonian is given for these subsystems by

$$\mathcal{H}_-^{\mathrm{red}}(x_\pm, \pi_\pm) = \pi_\pm^2 - \exp(2x_\pm). \tag{7.9}$$

All trajectories of these subsystems are singular, since in fact they reach infinity at finite time. But there is no singularity at all in the full reduced system $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})_-$, whose trajectories are simply the connected components of the curves obtained by intersecting $\mathcal{S}_-(2) \subset \mathbb{R}^3$ with the planes of fixed energy $u_2 = E$ for any constant E. These curves are hyperbolae or ellipses depending on the sign of E,

$$u_3^2 - E u_4^2 = 1. \tag{7.10}$$

At negative energy $E < 0$ the trajectory is periodic. The corresponding closed geodesic on $SL(2, \mathbb{R})$ connects the two components of the big cell, $\mathrm{sign}(u_4) = \pm$, through the lower dimensional submanifold, $u_4 = 0$. In effect, the two singular Toda lattices are regularized by their trajectories being glued together "at infinity". This illustrates the regularization obtained for general n as explained at the end of Section 3. It is worth noting that the surface $\mathcal{S}_-(2)$ is *not simply connected*. In fact, the loop given by, say, the ellipse in (7.10) for some $E < 0$ cannot be contracted to a point on the surface.

7.2. *The case of* $SL(3, \mathbb{R})$

We next analyse the structure of the reduced phase space then discuss the integration of the equation of motion.

7.2.1. The reduced phase space for $SL(3,\mathbb{R})$

We can choose $I_- = I_-^+, I_+ = I_+^+$ without loss of generality since all choices of (I_-, I_+) are equivalent for $SL(n,\mathbb{R})$ with odd n. Then the general element $(g, J) \in \mathcal{M}^{\mathrm{DS}}$ can be parametrized as

$$J = \begin{pmatrix} 0 & u_2 & u_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} (u_2u_4 + u_3u_5) & (u_2u_5 + u_3u_6) & u_4 \\ (u_2u_5 + u_3u_6) & u_4 & u_5 \\ u_4 & u_5 & u_6 \end{pmatrix}. \tag{7.11}$$

In order to prove the regularity of the hypersurface $\mathcal{S}_+(3) \subset \mathbb{R}^5$ defined by the equation

$$P(u) := \det g(u) = 1, \tag{7.12}$$

we now verify that $\mathrm{d}P$ does not vanish at any point of $\mathcal{S}_+(3)$. For this it will be enough to inspect the vanishing of the first three components of $\mathrm{d}P$,

$$\begin{aligned} \frac{\partial P}{\partial u_2} &= (u_4^2u_6 + u_4u_5^2) - 2u_5u_6(u_2u_5 + u_3u_6) = 0, \\ \frac{\partial P}{\partial u_3} &= (3u_4u_5u_6 - u_5^3) - 2u_6^2(u_2u_5 + u_3u_6) = 0, \\ \frac{\partial P}{\partial u_4} &= u_2(2u_4u_6 + u_5^2) + u_3(3u_5u_6) - 3u_4^2 = 0. \end{aligned} \tag{7.13}$$

To show that $\mathrm{d}P = 0$ and $P = 1$ are not compatible, we proceed by distinguishing various cases. First we look at the case where we assume that

$$u_5u_6 \neq 0. \tag{7.14}$$

Then we can combine the first two equations in (7.13) to yield

$$u_4 = \frac{u_5^2}{u_6}. \tag{7.15}$$

Substituting this back into the first equation in (7.13) gives

$$u_2 = \frac{u_5^2}{u_6^2} - u_3\frac{u_6}{u_5}. \tag{7.16}$$

Actually, the third equation in (7.13) is then also satisfied. However, if we now plug back (7.15), (7.16) into (7.12) we find that the determinant vanishes since its rows are all proportional with each other. Therefore we have proved that (7.14) is not compatible with the requirement that $P = 1$ and $\mathrm{d}P = 0$. So a "singularity" can only occur at a point where (7.14) does not hold. There are then three further subcases, such as

$$u_5 = u_6 = 0 \quad \text{or} \quad u_6 = 0 \quad \text{or} \quad u_5 = 0. \tag{7.17}$$

The reader can easily check that these are all excluded by (7.12) and (7.13). In conclusion, we have proved that $\mathcal{S}_+(3) \subset \mathbb{R}^3$ is a *regular hypersurface*. This implies that $\mathcal{S}_+(3)$ is *diffeomorphic* to $\mathcal{M}^{\mathrm{red}}$ as was explained in Section 6.

We know from Proposition 2 that the reduced system $(\mathcal{M}^{\text{red}}, \Omega^{\text{red}}, \mathcal{H}^{\text{red}})$ contains the Toda lattice with good sign. For $(I_-, I_+) = (I_-^+, I_+^+)$, using the Bruhat decomposition in (3.17), this Toda lattice lives on the open submanifold $\mathcal{M}_e^{\text{red}} \subset \mathcal{M}^{\text{red}}$. To identify $\mathcal{M}_e^{\text{red}}$ in terms of the double DS gauge in (7.11), we consider the mapping

$$Q := (Q_2, Q_3) : \mathcal{M}^{\text{red}} \to \mathbb{R} \times \mathbb{R}, \quad Q_3(g, J) = u_6, \quad Q_2(g, J) = \det \begin{pmatrix} u_4 & u_5 \\ u_5 & u_6 \end{pmatrix}. \tag{7.18}$$

Since g is Gauss decomposable if and only if its principal minors are positive, we have

$$\mathcal{M}_e^{\text{red}} = Q^{-1}\left(\mathbb{R}^+ \times \mathbb{R}^+\right). \tag{7.19}$$

Proposition 5. *The phase space $\mathcal{M}_e^{\text{red}}$ of the $SL(3, \mathbb{R})$ Toda lattice with good sign is a connected component of $\mathcal{M}^{\text{red}}$, i.e., it is disconnected from its complement.*

Proof. The claim that the open submanifold $\mathcal{M}_e^{\text{red}} \subset \mathcal{M}^{\text{red}}$ is disconnected from its complement follows if for some arbitrary $t_0 \in \mathbb{R}$ and $\epsilon > 0$ we demonstrate the *non-existence* of a continuous path, $\gamma(t) = (g(t), J(t)) \in \mathcal{M}^{\text{red}}$ for $t \in [t_0, t_0 + \epsilon]$, such that

$$Q_2(\gamma(t)) > 0, \quad Q_3(\gamma(t)) > 0 \qquad \text{for} \quad t > t_0 \tag{7.20}$$

and

$$Q_2(\gamma(t_0)) Q_3(\gamma(t_0)) = 0. \tag{7.21}$$

To give an indirect proof, suppose that the statement is not true. Then there exists a continuous path $\gamma(t)$ satisfying (7.20) and (7.21). Let us assume for the moment that (7.20) and (7.21) imply

$$Q_2(\gamma(t_0)) = Q_3(\gamma(t_0)) = 0. \tag{7.22}$$

Then we see from (7.18) that

$$u_5(t_0) = 0. \tag{7.23}$$

Looking at $g(u)$ in (7.11) at $t = t_0$, we also see that the condition $\det g(u) = 1$ requires

$$u_4(t_0) = -1. \tag{7.24}$$

However, (7.20) and the formula of Q_2 imply that

$$u_4(t) > 0 \quad \text{for}\, t > t_0. \tag{7.25}$$

If the curve $\gamma(t)$ is continuous then $u_4(t)$ is a continuous function at $t = t_0$, which is impossible by the above two equations.

To complete the proof, it remains to show (7.22). Obviously, the other possibilities allowed by (7.21) are

$$Q_3(\gamma(t_0)) = 0 \quad \text{and} \quad Q_2(\gamma(t_0)) > 0 \tag{7.26}$$

or

$$Q_3(\gamma(t_0)) > 0 \quad \text{and} \quad Q_2(\gamma(t_0)) = 0. \tag{7.27}$$

The first possibility (7.26) can be denied at once since it is inconsistent with the definitions for Q_2 and Q_3 (cf.(7.18)). To deny (7.27), we notice that, according to Proposition 4, the involutive symmetry $\psi_{(D_+,D_+)}$ would send a curve satisfying (7.20) and (7.27) into one satisfying (7.20) and (7.26), because in the $SL(3,\mathbb{R})$ case $\psi_{(D_+,D_+)}$ interchanges the minors Q_2 and Q_3. Alternatively, we may use $0 = Q_2(\gamma(t_0)) = u_4(t_0)u_6(t_0) - u_5^2(t_0)$ to obtain

$$\begin{aligned} 0 < Q_3^3(\gamma(t_0)) \det g(t_0) &= u_6^3(t_0) \det \begin{pmatrix} g_{11} & g_{12} & u_4 \\ g_{12} & u_4 & u_5 \\ u_4 & u_5 & u_6 \end{pmatrix} (t_0) \\ &= -\{u_5^3(t_0) - g_{12}(t_0)u_6^2(t_0)\}^2, \end{aligned} \tag{7.28}$$

which is, again, a contradiction. Thus (7.22) holds and the proof is complete. □

Remark 5. Notice from the above proof that the boundary of the positive quadrant of $\mathbb{R}\times\mathbb{R}$ does not belong to the image of the map $(Q_2, Q_3) : \mathcal{M}^{\text{red}} \to \mathbb{R}\times\mathbb{R}$ induced by the principal minors of $g \in SL(3,\mathbb{R})$.

7.2.2. Trajectories of the reduced system for $SL(3,\mathbb{R})$.

In addition to the connected component $\mathcal{M}_e^{\text{red}}$, the system $(\mathcal{M}^{\text{red}}, \Omega^{\text{red}}, \mathcal{H}^{\text{red}})$ contains three Toda lattices for which the Hamiltonian is not bounded from below. We below exhibit trajectories of the system that connect all these three Toda lattices. This is a non-trivial illustration of the fact that the Hamiltonian vector field is incomplete for every Toda lattice with "bad signs".

We wish to find the trajectory of the reduced system associated with some initial value $(u_2,\dots,u_6)$ at $t=0$ parametrizing a point $(g, J) \in \mathcal{M}^{\text{DS}} = \mathcal{M}^{\text{red}}$. As was explained in Remark 3, for this it is enough to solve the linear equation with constant coefficients in (6.15b), which now reads

$$\frac{\mathrm{d}^3u_6(t)}{\mathrm{d}t^3} - u_2\frac{\mathrm{d}u_6(t)}{\mathrm{d}t} - u_3u_6(t) = 0. \tag{7.29}$$

For simplicity, we choose to consider only those trajectories for which the characteristic equation corresponding to (7.29),

$$y^3 - u_2y - u_3 = \det(y\mathbf{1}_3 - J(u_2,u_3)) = 0, \tag{7.30}$$

has roots of the form

$$y_1 = -2a, \quad y_2 = a + b\sqrt{-1}, \quad y_3 = a - b\sqrt{-1}, \qquad a, b \in \mathbb{R}\setminus\{0\}. \tag{7.31}$$

We note that this excludes the initial values in $\mathcal{M}_e^{\text{red}}$, since the Lax matrix of the Toda lattice with good sign has distinct *real* eigenvalues.

The existence of a pair of complex conjugate roots of (7.30) may be ensured, for instance, by choosing

$$u_2 < 0, \qquad u_3 \neq 0. \tag{7.32}$$

In fact, if (7.32) holds, then introducing r and ϑ by

$$r := -\mathrm{sign}(u_3)\sqrt{\left|\frac{u_2}{3}\right|}, \qquad \sinh\vartheta := -\frac{1}{2}\frac{u_3}{r^3}, \tag{7.33a}$$

one has

$$a = r\sinh\frac{\vartheta}{3}, \qquad b = \sqrt{3}r\cosh\frac{\vartheta}{3}. \tag{7.33b}$$

The solution of (7.29) corresponding to roots of the form in (7.31) can be written as

$$u_6(t) = A\mathrm{e}^{-2at} + \mathrm{e}^{at}\left(B\sin bt + C\cos bt\right), \tag{7.34}$$

where A, B, C are real constants determined by the initial condition. We shall not need the explicit form of these constants, only that for a generic initial condition they do not vanish, which is obvious. According to (6.15a), the complete solution is then given by

$$u_2(t) = u_2, \quad u_3(t) = u_3, \quad u_4(t) = \frac{\mathrm{d}^2 u_6(t)}{\mathrm{d}t^2}, \qquad u_5(t) = \frac{\mathrm{d}u_6(t)}{\mathrm{d}t}. \tag{7.35}$$

We are interested in the qualitative behaviour of the principal minors $Q_3(u(t))$, $Q_2(u(t))$ of $g(u(t))$ along the above trajectory. We have $Q_3(u(t)) = u_6(t)$ and from (7.34), (7.35) we can determine $Q_2(u(t))$ as

$$Q_2(u(t)) = u_6(t)u_4(t) - u_5^2(t) = \hat{A}\mathrm{e}^{2at} + \mathrm{e}^{-at}\left(\hat{B}\sin bt + \hat{C}\cos bt\right). \tag{7.36}$$

The computation yields

$$\hat{A} = -2b^2(B^2 + C^2). \tag{7.37}$$

The explicit form of $\hat{B}$, $\hat{C}$ will not be needed.

It is important that the coefficient $\hat{A}$ of e^{2at} in (7.36) is *negative*. Now we argue that the coefficient A of e^{-2at} in (7.34) must also be *negative*. Indeed, if A was positive, then $Q_3(u(t))$ would have the limit $+\infty$ as t tends to $-(\mathrm{sign}(a))\,\infty$. On the other hand, since $\hat{A}$ is negative, $Q_2(u(t))$ oscillates around zero as t tends to $-(\mathrm{sign}(a))\,\infty$. But then the trajectory would necessarily enter $\mathcal{M}_e^{\mathrm{red}}$, where both Q_2 and Q_3 are positive. This is impossible since the initial value of the trajectory (at $t = 0$) is outside $\mathcal{M}_e^{\mathrm{red}}$, which is disconnected from its complement. This means that $A < 0$ must necessarily hold for the generic solution belonging to characteristic eigenvalues of the form in (7.31). (Actually, without using (7.37), an extension of this argument would in itself imply that both A and $\hat{A}$ must be negative.)

We have seen that the trajectory associated with a generic initial condition for which $J(u_2, u_3)$ has eigenvalues of the form in (7.31) satisfies (7.34), (7.36) with $A < 0$, $\hat{A} < 0$ and non-zero B, C, $\hat{B}$, $\hat{C}$. This implies the following alternatives for the asymptotic behaviour of such a trajectory:

(a) If $a < 0$, then

$$\lim_{t\to+\infty} Q_3(u(t)) = -\infty, \qquad \lim_{t\to-\infty} Q_2(u(t)) = -\infty, \tag{7.38a}$$

and $Q_3(u(t))$ (resp. $Q_2(u(t))$) oscillates around 0 for large negative (resp. positive) time.

(b) If $a > 0$, then

$$\lim_{t\to-\infty} Q_3(u(t)) = -\infty, \qquad \lim_{t\to+\infty} Q_2(u(t)) = -\infty, \tag{7.38b}$$

and $Q_3(u(t))$ (resp. $Q_2(u(t))$) oscillates around 0 for large positive (resp. negative) time.

In either case, the trajectory oscillates between a pair of connected components of the big cell of $SL(3,\mathbb{R})$ as $|t|$ tends to ∞. These pairs of connected components consist of the determinant one matrices for which one of the principal minors is negative. The pair in question is different for t approaching plus or minus infinity. It follows that the trajectory enters all the three open submanifolds $\mathcal{M}^{\mathrm{red}}_m \subset \mathcal{M}^{\mathrm{red}}$ for $m \neq e$ induced by the Bruhat decomposition. In particular, this confirms that the Hamiltonian vector field of the Toda lattice $(\mathcal{M}^{\mathrm{red}}_m, \Omega^{\mathrm{red}}_m, \mathcal{H}^{\mathrm{red}}_m)$ is incomplete for $m \neq e$.

Recall that J appearing in $(g, J) \in \mathcal{M}^{\mathrm{DS}}$ is conjugate to the Lax matrix of a Toda lattice if g belongs to the big cell. In conclusion, the qualitative behaviour found above is consistent with the general result that the trajectory of the Toda lattice blows up if the Lax matrix comprising the initial data admits a complex eigenvalue [GS,KY].

8. Conclusion

In this paper we investigated the natural regularization of incomplete Toda lattices associated with $sl(n,\mathbb{R})$ that results from Hamiltonian reduction. We obtained a reduced phase space from $T^*SL(n,\mathbb{R})$ which contains an open dense submanifold consisting of 2^{n-1} Toda lattices and a complementary part consisting of lower dimensional submanifolds serving to glue together the blowing up trajectories of the Toda lattices. We developed tools, such as the double DS gauge and the hypersurface model of $\mathcal{M}^{\mathrm{red}}$, for further investigating the global structure of the reduced system, and used them to analyse the simplest examples corresponding to $SL(n,\mathbb{R})$ for $n = 2, 3, 4$ (see also Appendix A). The results presented in the main text for the Lie algebra $sl(n,\mathbb{R})$ can be generalized to an arbitrary simple Lie algebra as explained in Appendix B.

Much work remains to be done to explore the structure of the reduced system, which appears quite interesting, and complicated. For example, the Toda lattice with good sign, whose flow is complete, should occupy a connected component of the phase space whenever it is contained as a subsystem in the reduced system, although this conjecture has been proven for $n = 2, 3, 4$ only. It seems clear, but has not been shown yet, that the reduced phase space is not the cotangent bundle of some configuration space in general. Concerning the hypersurface model of $\mathcal{M}^{\mathrm{red}}$, the regularity conjectures given in (6.18) and (6.25) should be verified for arbitrary n.

A particularly challenging problem is to construct the quantum mechanical version of the reduced system and to determine, e.g., the joint spectra of the operators corresponding to the Casimirs of $sl(n, \mathbb{R})$. The quantization of the open Toda lattice with good sign is surveyed in [STS] using a reduction method based on the Iwasawa decomposition. One might try to extend this "first quantize then reduce" method to our case based on the Bruhat decomposition, or one might try to directly quantize the reduced classical system. Direct quantization has been attempted in [F,KT] for the simplest case of $SL(2, \mathbb{R})$.

It could be also worthwhile to search for a classification of singular solutions in the field theoretic version of the open Toda lattices with the aid of the analogue of the Hamiltonian reduction used here, which is described in [FORTW]. By direct methods, such classification was investigated in [PP] for the Liouville equation related to $SL(2, \mathbb{R})$.

Finally, let us remind that singular solutions of many integrable systems arise from the breakdown of solvability in the factorization problem inherent in the AKS scheme, and that Hamiltonian reduction can provide a regularization of such singularities in many instances (see [RSTS1,R,RSTS2]). In the present study we demonstrated that the reduced Hamiltonian systems realizing the regularization of singular Toda lattices possess a rich and intriguing structure. We hope that our paper may serve as a non-trivial illustration of the aforementioned general properties of integrable systems.

Appendix A. $\mathcal{M}^{\rm red}_{\pm e}$ are connected components of $\mathcal{M}^{\rm red}(I_-^+, I_+^+)$ for $SL(4, \mathbb{R})$

In Section 7 we discussed in some detail the reduced systems for the two simplest examples $SL(2, \mathbb{R})$ and $SL(3, \mathbb{R})$. If we go one step further, i.e., to $SL(4, \mathbb{R})$, we learn from Proposition 1 that two non-isomorphic reduced systems $(\mathcal{M}^{\rm red}, \Omega^{\rm red}, \mathcal{H}^{\rm red})_\pm$ are possible depending on which orbit of Δ the pair (I_-, I_+) belongs to. Proposition 2 then says that the Toda lattice with good sign (i.e., all $\mu_i > 0$ in (4.1)) appears only in the reduced system $(\mathcal{M}^{\rm red}, \Omega^{\rm red}, \mathcal{H}^{\rm red})_+$, and that it appears in two copies among the eight Toda lattices allowed. The aim of this appendix is to show that these Toda lattices with good sign, which arise from the components of the big cell of $SL(4, \mathbb{R})$ containing $\pm e$, are connected components of the reduced phase space. This will provide a proof of the conjecture for $n = 4$ mentioned at the beginning of Section 7. Our strategy will be similar to the one used for $SL(3, \mathbb{R})$.

We first recall that the double DS gauge introduced in Section 6 allows for parametrizing the general element $(g, J) \in \mathcal{M}^{\rm red}(I_-^+, I_+^+)$ as

$$J = \begin{pmatrix} 0 & u_2 & u_3 & u_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} & u_5 \\ g_{1,2} & g_{1,3} & u_5 & u_6 \\ g_{1,3} & u_5 & u_6 & u_7 \\ u_5 & u_6 & u_7 & u_8 \end{pmatrix}, \tag{A.1a}$$

where

$$g_{1,1} = u_2(u_2u_6 + u_3u_7 + u_4u_8) + u_3u_5 + u_4u_6,$$

$$g_{1,2} = u_2u_5 + u_3u_6 + u_4u_7, \qquad \text{(A.1b)}$$
$$g_{1,3} = u_2u_6 + u_3u_7 + u_4u_8.$$

These parameters $u = (u_2, \dots, u_8)$ are subject to the condition $P(u) = \det g(u) = 1$. The set of globally defined functions in (5.15)

$$Q_4(g, J) = u_8, \qquad Q_3(g, J) = \det \begin{pmatrix} u_6 & u_7 \\ u_7 & u_8 \end{pmatrix},$$
$$Q_2(g, J) = \det \begin{pmatrix} g_{1,3} & u_5 & u_6 \\ u_5 & u_6 & u_7 \\ u_6 & u_7 & u_8 \end{pmatrix}, \qquad \text{(A.2)}$$

then give the mapping $Q := (Q_2, Q_3, Q_4) : \mathcal{M}^{\mathrm{red}} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Using this we can write the open submanifolds $\mathcal{M}^{\mathrm{red}}_{\pm e} \subset \mathcal{M}^{\mathrm{red}}$, each of which carries the Toda lattice with good sign, as

$$\mathcal{M}^{\mathrm{red}}_{\pm e} = Q^{-1}\left(\mathbb{R}^{\pm} \times \mathbb{R}^{\pm} \times \mathbb{R}^{\pm}\right). \qquad \text{(A.3)}$$

To see that one of the submanifolds, say $\mathcal{M}^{\mathrm{red}}_{e}$, is topologically disconnected from its complement $\mathcal{M}^{\mathrm{red}} \setminus \mathcal{M}^{\mathrm{red}}_{e}$, let us first assume that this is not true. Then we can consider a continuous path $\gamma(t) = (g(t), J(t)) \in \mathcal{M}^{\mathrm{red}}$ for $t \in [t_0, t_0 + \epsilon]$ ($t_0 \in \mathbb{R}$ and $\epsilon > 0$) connecting one point from $\mathcal{M}^{\mathrm{red}}_{e}$ and another from $\left(\mathcal{M}^{\mathrm{red}} \setminus \mathcal{M}^{\mathrm{red}}_{e}\right) \cap \mathcal{M}^{\mathrm{red}}_{\mathrm{low}}$ in such a way that

$$Q_2(\gamma(t)) > 0, \quad Q_3(\gamma(t)) > 0, \quad Q_4(\gamma(t)) > 0 \quad \text{for} \quad t > t_0 \qquad \text{(A.4)}$$

and

$$Q_2(\gamma(t_0))Q_3(\gamma(t_0))Q_4(\gamma(t_0)) = 0. \qquad \text{(A.5)}$$

In what follows we shall prove that in fact there exists no path $\gamma(t)$ satisfying (A.4) and (A.5), which implies that our assumption is wrong and hence $\mathcal{M}^{\mathrm{red}}_{e}$ must be disconnected from its complement. Our proof consists of two parts. In the first part, we show that (A.4) and (A.5) actually imply

$$Q_2(\gamma(t_0)) = Q_3(\gamma(t_0)) = Q_4(\gamma(t_0)) = 0. \qquad \text{(A.6)}$$

In the second part we shall show that (A.4) and (A.6) lead to a contradiction.

We begin the first part of the proof by noting that (A.4) and (A.5) imply either

$$Q_4(\gamma(t_0)) = 0, \qquad Q_3(\gamma(t_0)) \geq 0, \qquad Q_2(\gamma(t_0)) \geq 0 \qquad \text{(A.7a)}$$

or

$$Q_4(\gamma(t_0)) \geq 0, \qquad Q_3(\gamma(t_0)) = 0, \qquad Q_2(\gamma(t_0)) \geq 0 \qquad \text{(A.7b)}$$

or the case where Q_2 and Q_4 are interchanged in (A.7a). However, it is enough to consider only the two cases (A.7a), (A.7b) since one can convert the last case into (A.7a) using the involutive symmetry transformation presented in Section 5 (see Proposition 4).

Now consider the case (A.7a). Since

$$0 = Q_4(\gamma(t_0)) = u_8(t_0), \tag{A.8}$$

we have $Q_3(\gamma(t_0)) = -u_7^2(t_0) \leq 0$. From (A.7a) we find $u_7(t_0) = 0$ and

$$Q_3(\gamma(t_0)) = 0. \tag{A.9}$$

It then follows that $Q_2(\gamma(t_0)) = -u_6^3(t_0)$ and therefore from (A.7a) that

$$u_6(t_0) \leq 0. \tag{A.10}$$

On the other hand, at $t > t_0$, condition (A.4) requires

$$0 < Q_4(\gamma(t)) = u_8(t) \quad \text{and} \quad 0 < Q_3(\gamma(t)) = u_6(t)u_8(t) - u_7^2(t)\,, \tag{A.11}$$

and hence

$$u_6(t) > 0. \tag{A.12}$$

Due to the continuity of the path this is consistent with (A.10) only if $u_6(t_0) = 0$, i.e.,

$$Q_2(\gamma(t_0)) = 0 \tag{A.13}$$

as well. We have thus learned that (A.7a) reduces to (A.6).

Next, consider the case (A.7b) for which

$$0 = Q_3(\gamma(t_0)) = u_6(t_0)u_8(t_0) - u_7^2(t_0). \tag{A.14}$$

From this we find

$$\begin{aligned} Q_4^3(\gamma(t_0))Q_2(\gamma(t_0)) &= u_8^3(t_0)\ \det \begin{pmatrix} g_{1,3} & u_5 & u_6 \\ u_5 & u_6 & u_7 \\ u_6 & u_7 & u_8 \end{pmatrix} (t_0) \\ &= -\{u_7^3(t_0) - u_5(t_0)u_8^2(t_0)\}^2 \leq 0. \end{aligned} \tag{A.15}$$

On account of (A.7b) the equality must hold in (A.15) and, accordingly, either $Q_4(\gamma(t_0))$ or $Q_2(\gamma(t_0))$ must be zero. This allows us to go back to the previous case (A.7a) (or the last case which is equivalent to (A.7a) due to the involutive symmetry), and thus we conclude that (A.7b) also reduces to (A.6). This completes the first part of our proof.

We here remind that on a smooth manifold every continuous path between two fixed points is continuously deformable to a smooth (that is C^∞) path. In particular, if there is a continuous path on $\mathcal{M}^{\text{red}}$ satisfying (A.4) and (A.5), then there also exists a smooth path meeting these conditions.

For the second part of the proof we need to introduce some notations. First, given a C^∞ function $f(t)$ of $t \in [t_0, t_0 + \epsilon]$, let $\mathcal{L}(f)$ be the leading term of the function f in the formal Taylor expansion at $t = t_0$. That is, if $(\mathrm{d}^i f/\mathrm{d}t^i)(t_0) \neq 0$ for some $i \geq 0$,

$$\mathcal{L}(f) := \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t_0)\,(t - t_0)^n, \tag{A.16a}$$

where

$$n = \deg(f) := \min\left\{ i \,\middle|\, \frac{\mathrm{d}^i f}{\mathrm{d}t^i}(t_0) \neq 0 \right\}. \tag{A.16b}$$

We put $\deg(f) = \infty$ and $\mathcal{L}(f) = 0$ if $(\mathrm{d}^i f/\mathrm{d}t^i)(t_0) = 0$ for all i. It is obvious that

$$\deg(fg) = \deg(f) + \deg(g), \qquad \mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g) \tag{A.17}$$

for arbitrary smooth functions f, g. If f is written as a sum of smooth functions, $f = \sum_{i=1}^m f_i$, then we define

$$\mathcal{T}(f \mid f_1, \ldots, f_m) := \sum_{i=1}^m \delta_{\sigma(i),k}\mathcal{L}(f_i), \tag{A.18a}$$

where

$$\sigma(i) := \deg(f_i) \quad \text{and} \quad k := \min\{\deg(f_1), \ldots, \deg(f_m)\}. \tag{A.18b}$$

Note that

$$a\,\mathcal{T}(f \mid f_1, \ldots, f_m) = \mathcal{T}(af \mid af_1, \ldots, af_m) \quad \forall a \in \mathbb{R}. \tag{A.19}$$

It should be stressed that $\mathcal{T}(f \mid f_1, \ldots, f_m)$ is defined with respect to the set of functions $\{f_1, \ldots, f_m\}$, and in general the outcome of the operation depends on the set. [5] Upon specifying the set, we may write $\mathcal{T}(f_1 + \cdots + f_m)$ or simply $\mathcal{T}(f)$ for $\mathcal{T}(f \mid f_1, \ldots, f_m)$. From the definitions, it is easy to show that

$$\begin{aligned} f(t) > 0 \quad \text{for all } t \in (t_0, t_0 + \epsilon] \;\Rightarrow\; & \mathcal{L}(f) \geq 0 \quad \text{and} \\ & \mathcal{T}(f \mid f_1, \ldots, f_m) \geq 0. \end{aligned} \tag{A.20}$$

Here $\mathcal{L}(f) = 0$ if and only if $\deg(f) = \infty$. Since the operation $\mathcal{T}(f \mid f_1, \ldots, f_m)$ in (A.18) keeps only the leading term(s) among the $\mathcal{L}(f_i)$ for $i = 1, \ldots, m$, $\mathcal{T}(f)$ vanishes if the non-zero leading terms cancel each other.

Let us now return to our reduced system. Recall that (A.6) implies $u_8(t_0) = u_7(t_0) = u_6(t_0) = 0$, and hence $u_5(t_0) = \pm 1$ from the condition $P(u(t_0)) = \det g(t_0) = 1$. On account of the fact that

$$\mathrm{d}P(u)|_{P(u)=1} = \pm(4\,\mathrm{d}u_5 - 2u_2\,\mathrm{d}u_7 - u_3\,\mathrm{d}u_8) \neq 0 \quad \text{at}\, u = u(t_0), \tag{A.21}$$

we see that the set $\{u_2, u_3, u_4, u_6, u_7, u_8\}$ can be taken as coordinates on a neighbourhood $\mathcal{U}$ of the point $\gamma(t_0)$ diffeomorphic to an open ball in $\mathbb{R}^6$. In $\mathcal{U} \subset \mathcal{M}^{\mathrm{red}}$, we can consider the *generic* class of smooth paths satisfying (A.4), (A.5) as well as the following additional condition for the coordinate functions along the path,

$$\mathcal{L}(u_i) \neq 0 \quad \text{for } i = 2, 3, 4, 6, 7, 8. \tag{A.22}$$

[5] For example, for $f = f_1 + f_2$ with $f_1 = (t - t_0) + (t - t_0)^2$ and $f_2 = -(t - t_0)$, we have $\mathcal{T}(f \mid f) = \mathcal{L}(f) = (t - t_0)^2$ whereas $\mathcal{T}(f \mid f_1, f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2) = 0$.

This class of paths is *not* empty, since, as is intuitively clear, every smooth path satisfying (A.4) and (A.5) can be deformed to a smooth path connecting the same end points in such a way that (A.22) holds in addition to (A.4) and (A.5). The class of paths satisfying (A.22) is generic among the smooth paths satisfying (A.4) and (A.5), because (A.22) is stable under small deformations.

Let now $\gamma(t)$ be an arbitrary smooth path subject to the requirements (A.4), (A.5) and (A.22). If we write $u_5(t) = \pm 1 + \hat{u}_5(t)$, then the functions $u_8(t)$, $u_7(t)$, $u_6(t)$, $\hat{u}_5(t)$, $g_{1,3}(t)$ all vanish at $t = t_0$, and hence the degree defined in (A.16b) is non-zero for each of these functions. In addition, the degrees of $u_6(t)$, $u_7(t)$, $u_8(t)$ are finite according to (A.22). In terms of these functions the first condition in (A.4) reads

$$0 < Q_2(\gamma(t)) = -u_8 \pm 2u_6u_7 - u_6^3 + g_{1,3}u_6u_8 - g_{1,3}u_7^2 \mp 2\hat{u}_5u_8 + 2\hat{u}_5u_6u_7 - \hat{u}_5^2u_8. \tag{A.23}$$

For brevity we hereafter suppress the parameter t in the functions. Applying the operations (A.16) and (A.18) on the two inequalities in (A.11), respectively, we find

$$\mathcal{L}(u_8) > 0, \tag{A.24}$$

and

$$\mathcal{T}(Q_3|\, u_6u_8 - u_7^2) = \mathcal{T}(u_6u_8 - u_7^2) \geq 0, \tag{A.25}$$

where we used (A.20) and (A.22). From (A.25) we now notice that

$$\deg\,(u_7^2) \geq \deg\,(u_6u_8), \tag{A.26}$$

since otherwise $\mathcal{T}(u_6u_8 - u_7^2) = -\{\mathcal{L}(u_7)\}^2 < 0$. On the other hand, we see that, since $\deg\,(g_{1,3}) \geq 1$, the degree of any of the five terms in the second line of (A.23) is higher than the degree of at least one of the first three terms $\{-u_8, \pm 2u_6u_7, -u_6^3\}$. Indeed, the degrees of those three terms in the second line that have the factor u_8 are obviously higher than that of u_8, while $\deg\,(g_{1,3}u_7^2) \geq \deg\,(g_{1,3}u_6u_8) > \deg\,(u_8)$ and $\deg\,(\hat{u}_5u_6u_7) > \deg\,(u_6u_7)$. Thus, applying (A.18) on (A.23) and using (A.20) we obtain that

$$\mathcal{T}(Q_2) = \mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) \geq 0. \tag{A.27}$$

To be more explicit, let us put

$$\mathcal{L}(u_6) = a(t - t_0)^m, \qquad \mathcal{L}(u_7) = b(t - t_0)^n, \qquad \mathcal{L}(u_8) = c(t - t_0)^l, \tag{A.28}$$

where a, b, c are non-vanishing constants and m, n, l are some positive integers. Observe then that (A.12) and (A.22) imply $\mathcal{L}(u_6) > 0$, that is

$$a > 0, \tag{A.29}$$

and therefore from (A.19) and (A.27) we have

$$0 \leq a\,\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = -\mathcal{T}\big(ac(t - t_0)^l \mp 2a^2b(t - t_0)^{m+n} + a^4(t - t_0)^{3m}\big). \tag{A.30}$$

Observe also that (A.24) is equivalent to

$$c > 0, \tag{A.31}$$

and that (A.26) implies

$$m + l \leq 2n. \tag{A.32}$$

Now suppose $m + n < 3m$. From (A.32) we then have $l < m + n$ and hence

$$a\,\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = -ac(t - t_0)^l < 0, \tag{A.33}$$

which contradicts with (A.30). Thus we find

$$m + n \geq 3m. \tag{A.34}$$

However, $m + n > 3m$ is impossible since it leads to either (A.33) for $l < 3m$, or

$$a\,\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = -a^4(t - t_0)^{3m} < 0, \tag{A.35}$$

for $l > 3m$, or

$$a\,\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = -(ac + a^4)(t - t_0)^{3m} < 0, \tag{A.36}$$

for $l = 3m$, all of which contradict with (A.30). Thus we must have $m + n = 3m$, and combining this with (A.32) we get $l \leq 3m$. But since $l < 3m$ leads again to (A.33) we conclude that the three terms in (A.30) are of the same degree, $l = m + n = 3m$, i.e.,

$$l = 3m \quad \text{and} \quad n = 2m. \tag{A.37}$$

Having found the ratio of the three degrees, we see that (A.25) reads

$$\mathcal{T}(u_6u_8 - u_7^2) = (ac - b^2)(t - t_0)^{4m} \geq 0. \tag{A.38}$$

It then follows that

$$a\,\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = -\{(ac - b^2) + (b \mp a^2)^2\}(t - t_0)^{3m} \leq 0. \tag{A.39}$$

Comparing this with (A.30) we find that the equality must hold, and accordingly

$$\mathcal{T}(-u_8 \pm 2u_6u_7 - u_6^3) = 0. \tag{A.40}$$

Thus the leading terms of the first three terms in (A.23) cancel each other, leaving higher degree terms. Therefore the degree of the function Q_2 in (A.23) necessarily satisfies

$$\deg(Q_2) > 3m. \tag{A.41}$$

On the other hand, we have

$$\deg(Q_4) = \deg(u_8) = 3m. \tag{A.42}$$

In the above we have proven that

$$\deg(Q_2) > \deg(Q_4) \tag{A.43}$$

for any smooth path $\gamma(t)$ satisfying (A.4), (A.5) and (A.22). However, this is in conflict with the involutive symmetry $\hat{\psi}_{(D_+,D_+)} : \mathcal{M}^{\rm red} \to \mathcal{M}^{\rm red}$ described in Section 5, which shows that if $\gamma(t)$ is a path connecting $\mathcal{M}_e^{\rm red}$ to its complement then there exists also another path $\hat{\gamma}(t)$, given by $\hat{\gamma}(t) = \hat{\psi}_{(D_+,D_+)}(\gamma(t))$, that does the same in such a way that $Q_i(\hat{\gamma}(t)) = Q_{4+i-2}(\gamma(t))$. Applying this symmetry to a smooth path satisfying (A.4), (A.5), (A.22) and $\deg(Q_2) > \deg(Q_4)$ generically results in a smooth path satisfying (A.4), (A.5), (A.22) and $\deg(Q_2) < \deg(Q_4)$. This implies that there must also exist a smooth path satisfying (A.4), (A.5), (A.22) and $\deg(Q_2) < \deg(Q_4)$. But this is impossible since we proved that (A.4), (A.5) and (A.22) imply (A.43). We conclude from this contradiction that $\mathcal{M}_e^{\rm red}$ is indeed disconnected from its complement.

Having shown that $\mathcal{M}_e^{\rm red}$ is a connected component of $\mathcal{M}^{\rm red}$, the symmetry given in (5.14) shows that $\mathcal{M}_{-e}^{\rm red}$ is also a connected component.

We expect that the Toda lattice with good sign is realized as a connected component whenever it is contained in the reduced phase space associated with $SL(n,\mathbb{R})$, for any n. However, a new idea is required for proving this conjecture, since the method used above would become impractical for $n > 4$.

Appendix B. Generalization for arbitrary simple Lie algebras

The framework for regularizing Toda lattices with "bad signs" described in the main text for $sl(n,\mathbb{R})$ can be straightforwardly generalized for an arbitrary simple Lie algebra. We now briefly present this generalization.

Let $\mathcal{G}$ be the normal (split) real form of some complex simple Lie algebra. Then $\mathcal{G}$ is generated by the Chevalley generators

$$h_{\alpha_i},\quad e_{\alpha_i},\quad e_{-\alpha_i} \tag{B.1}$$

associated with the simple roots α_i for $i = 1, \ldots, r := \mathrm{rank}(\mathcal{G})$. The Chevalley involution θ of $\mathcal{G}$ operates as

$$\theta(h_{\alpha_i}) = -h_{\alpha_i}, \qquad \theta(e_{\alpha_i}) = -e_{-\alpha_i}. \tag{B.2}$$

We have $\mathrm{tr}\left(e_{\alpha_i} e_{-\alpha_i}\right) > 0$ with "tr" being the Killing form of $\mathcal{G}$ up to a positive constant. In the triangular decomposition of Eq. (2.1) now $\mathcal{G}_0 = \mathrm{span}\{h_{\alpha_i}\}_{i=1}^r$ is the splitting Cartan subalgebra of $\mathcal{G}$ and $\mathcal{G}_{>0}$ (resp. $\mathcal{G}_{<0}$) are the subalgebras spanned by the positive (resp. negative) root vectors.

The phase space of the Toda lattice of our interest is $M_e = \mathcal{G}_0 \times \mathcal{G}_0 \simeq \mathcal{O}$ given as in (2.18), now using

$$I_+ = \sum_{i=1}^r \nu_i^+ e_{\alpha_i}, \quad I_- = \sum_{i=1}^r \nu_i^- e_{-\alpha_i}, \qquad \nu_i^\pm \neq 0. \tag{B.3}$$

It has the symplectic form $\omega_e = \mathrm{d}\,\mathrm{tr}\,(p\,\mathrm{d}q)$ and, with $\nu_i := \nu_i^- \nu_i^+ \mathrm{tr}\left(e_{\alpha_i} e_{-\alpha_i}\right)$, the Hamiltonian

$$H_e(q,p) := \tfrac{1}{2}\mathrm{tr}(p^2) + \mathrm{tr}\left(I_- \mathrm{e}^q I_+ \mathrm{e}^{-q}\right) = \tfrac{1}{2}\mathrm{tr}(p^2) + \sum_{i=1}^{r} \nu_i \mathrm{e}^{\alpha_i(q)}. \tag{B.4}$$

The Toda lattice (M_e, ω_e, H_e), which is singular if $\nu_i < 0$ for some i, is contained in the reduced system following from a Hamiltonian reduction of the natural system $(\mathcal{M}, \Omega, \mathcal{H})$ on $\mathcal{M} = T^*G$ with G being a connected Lie group corresponding to $\mathcal{G}$. To define the reduction, N_+ and N_- in (3.8) are now taken to be the Lie subgroups of G associated with $\mathcal{G}_{>0}$ and $\mathcal{G}_{<0}$, respectively, the constrained manifold $\mathcal{M}^{\mathrm{c}}$ is defined similarly to (3.11), and $\mathcal{M}^{\mathrm{red}} = \mathcal{M}^{\mathrm{c}}/N$. We wish to sketch the structure of $(\mathcal{M}^{\mathrm{red}}, \Omega^{\mathrm{red}}, \mathcal{H}^{\mathrm{red}})$.

Let K be the Lie subgroup of G corresponding to the Lie subalgebra of $\mathcal{G}$ given by the fixed point set of θ. Let $\boldsymbol{M}^* \subset K$ be the *normalizer* and $\boldsymbol{M} \subset K$ the *centralizer* of $\mathcal{G}_0$ with respect to the adjoint representation of G on $\mathcal{G}$ restricted to $K \subset G$. Then $\boldsymbol{W} := \boldsymbol{M}^*/\boldsymbol{M}$ is the Weyl group of $\mathcal{G}$ with respect to the Cartan subalgebra $\mathcal{G}_0$. Denote by A the Lie subgroup of G corresponding to $\mathcal{G}_0$. We have (see [H,W]) the *Bruhat decomposition* of G:

$$G = \bigcup_{w \in \boldsymbol{W}} N_+ \boldsymbol{M} m_w^* A N_- \quad \text{(disjoint union)}, \tag{B.5}$$

where $m_w^* \in \boldsymbol{M}^*$ is an arbitrary representative of $w \in \boldsymbol{W}$. The big cell belongs to the identity element of $\boldsymbol{W}$. It is an open, dense submanifold in G having the connected components $G_m := N_+ m A N_-$ for all $m \in \boldsymbol{M}$, which are diffeomorphic to $N_+ \times A \times N_-$. The decomposition in (B.5) can be rewritten as

$$G = \bigcup_{m \in \boldsymbol{M}} G_m \cup G_{\mathrm{low}}, \qquad G_{\mathrm{low}} = \bigcup_{w \neq e} N_+ \boldsymbol{M} m_w^* A N_-. \tag{B.6}$$

The lower dimensional submanifold $N_+ \boldsymbol{M} m_w^* A N_- \subset G$ is diffeomorphic to the product of the factors on the left and right hand sides of the equality

$$N_+^w \boldsymbol{M} m_w^* A N_- = N_+ \boldsymbol{M} m_w^* A N_- = N_+ \boldsymbol{M} m_w^* A N_-^w \tag{B.7a}$$

with

$$N_+^w = N_+ \cap m_w^* N_+ (m_w^*)^{-1} \quad \text{and} \quad N_-^w = N_- \cap (m_w^*)^{-1} N_- m_w^*. \tag{B.7b}$$

In the same way as we saw in Section 3, the Bruhat decomposition of G induces a decomposition of $\mathcal{M}^{\mathrm{red}}$ of the form in (3.17). For $m = e$ the subsystem $(\mathcal{M}_m^{\mathrm{red}}, \Omega_m^{\mathrm{red}}, \mathcal{H}_m^{\mathrm{red}})$ is isomorphic to (M_e, Ω_e, H_e), otherwise it is a Toda lattice of the same kind obtained by simply replacing I_+ with $I_+^m = m I_+ m^{-1}$. In particular, the Hamiltonian $\mathcal{H}_m^{\mathrm{red}}$ of this Toda lattice has the form

$$\mathcal{H}_m^{\mathrm{red}}(q,p) := \tfrac{1}{2}\mathrm{tr}(p^2) + \mathrm{tr}\left(I_- \mathrm{e}^q I_+^m \mathrm{e}^{-q}\right) = \tfrac{1}{2}\mathrm{tr}(p^2) + \sum_{i=1}^{r} \mu_i \mathrm{e}^{\alpha_i(q)},$$
$$\mu_i = m_{\alpha_i} \nu_i, \tag{B.8}$$

where m_{α_i} is defined by $m e_{\alpha_i} m^{-1} = m_{\alpha_i} e_{\alpha_i}$. Therefore $\mathcal{M}^{\mathrm{red}}$ contains Toda lattices that are in general singular (incomplete) but are glued together in such a way that their singularities are regularized in the entire system. With the aid of Eqs. (6.1) and (6.2), one can introduce the double DS gauge $\mathcal{M}^{\mathrm{DS}} \subset \mathcal{M}^{\mathrm{c}}$ to provide a globally valid model of $\mathcal{M}^{\mathrm{red}}$, which could be useful for exploring the topology of $\mathcal{M}^{\mathrm{red}}$.

Remark 6. The analogue of the splitting in (2.22) is given by $\mathcal{G} = \mathcal{K} + \mathcal{G}_{\leq 0}$, where $\mathcal{K}$ is the Lie algebra of $K \subset G$. Thus one has an orbital model $\tilde{\mathcal{O}}$ of the Toda lattice with good sign similar to (2.27), where one must now take $I_- = -\theta(I_+)$. This Toda lattice arises as the reduced system from a Hamiltonian reduction of the system $(T^*G, \Omega, \mathcal{H})$ based on the symmetry group $K \times N_-$, whose action on T^*G is induced by the left translations of K on G and the right translations of N_- on G. The reduction is defined fixing the value of the momentum map of the K-action to zero, and fixing the momentum map of the N_--action to the value $I := (I_+ - \theta(I_-))$. Using the identifications $\mathcal{K}^* = (\mathcal{G}_{\leq 0})^{\perp}$ and $(\mathcal{G}_{\leq 0})^* = \mathcal{K}^{\perp}$, here $I \in \mathcal{K}^{\perp}$ represents a functional on $\mathcal{G}_{<0} \subset \mathcal{G}_{\leq 0}$. This treatment of the Toda lattice relates to the globally valid Iwasawa decomposition, $G = KAN_-$, of the group G. It reproduces the Hamiltonian reduction treatment given in [OP,P,FO] if one first performs the reduction of the K-symmetry, which leads to geodesic motion on the symmetric space $K\backslash G \simeq AN_-$.

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